

Foundations of the calculus of variations in generalized function algebras

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Abstract

We propose the use of algebras of generalized functions for the analysis of certain highly singular problems in the calculus of variations. After a general study of extremal problems on open subsets of Euclidean space in this setting we introduce the first and second variation of a variational problem. We then derive necessary (Euler-Lagrange equations) and sufficient conditions for extremals. The concept of association is used to obtain connections to a distributional description of singular variational problems. We study variational symmetries and derive an appropriate version of Nöther's theorem. Finally, a number of applications to geometry, mechanics, elastostatics and elastodynamics are presented.

Mathematics Subject Classification (2000): Primary: 49J27; secondary: 46F30, 49K27, 37K05

Keywords: singular variational problems, distributions, algebras of generalized functions, variational symmetries

1 Introduction

The study of singular problems in the calculus of variations has a long history and a sizeable literature on diverse aspects of this topic is available (cf., e.g., [15, 6, 46] and the literature cited therein). In this paper we introduce an approach to variational problems involving singularities based on the theory of algebras of generalized functions in the sense of Colombeau ([4, 5]). We are interested in extremizing functionals which are either singular (e.g., distributional) themselves or whose set of admissible variations includes generalized functions.

When considering singular variational problems in the above sense, distribution theory is only of limited use due to the nonlinear nature of the typical variational problems. The theory of algebras of generalized functions, on the other hand, provides a nonlinear extension of distribution theory which allows to model nonlinear singular problems while at the same time providing optimal consistency properties with respect to the linear theory. It has found an increasing number of applications in linear and nonlinear partial differential equations (cf., e.g., [36], or [35] for a recent survey), regularity theory and microlocal analysis (e.g., [10, 19, 7]), as well as in non-smooth differential geometry (e.g., [16, 28, 29, 31]). As a rather novel development, the order structure in Colombeau type algebras of generalized functions has been investigated in [1, 40, 32]. It allows the formulation of variational problems in the generalized setting and will therefore be crucial to this work.

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Furthermore, we study symmetry properties of variational problems, thereby continuing our investigations on Lie symmetries of differential equations in generalized functions ([26, 8, 23, 24]). Our aim here is to derive infinitesimal criteria and establish an appropriate version of Nöther's theorem.

The paper is organized as follows. After recalling some standard notations from symmetry group analysis and algebras of generalized functions, in Section 2 we examine extremal values of Colombeau generalized functions on open subsets of Euclidean space. As in the classical calculus of variations this will provide the basis for deriving necessary and sufficient conditions for general variational problems later on. Already at this stage, new effects occur due to the structural properties of the set of scalars in Colombeau theory: they form an ordered ring but not a field. The presence of zero divisors requires a more refined analysis than in the smooth case. In Section 3 we turn to calculus of variations proper. After singling out the appropriate spaces of generalized functions for formulating the problem, we study necessary and sufficient conditions for minimizers. In particular, we prove a fundamental lemma of the calculus of variations and obtain the Euler-Lagrange equations as necessary conditions. We also introduce the concept of minimizer in the sense of associations which allows to study variational problems on the distributional level. In section 4 we investigate variational symmetries in the framework of algebras of generalized functions. Here we derive an infinitesimal criterion and establish a Nöther theorem, allowing to derive conservation laws from variational symmetries. Finally, section 5 is devoted to a variety of applications, focussing on variational problems with singularities that require formulation in the Colombeau setting. The first example treats geodesics in a generalized Riemannian metric. Then we turn to particle mechanics with singular potentials. The third type of example is concerned with elasticity theory, either with degenerate material properties (loss of ellipticity) or with singular potentials: membranes with springs attached, beams with discontinuous or vanishing coefficients and rods with generalized stress-strain relation. In the last example we turn to the hyperbolic case: the wave equation with singular and nonlinear potential.

We now briefly introduce the notations that will be used throughout this paper. Our basic reference for the theory of algebras of generalized functions is [16]. For symmetry group analysis we refer to [42]. Concerning the calculus of variations, any standard text on the subject will cover what is needed in our approach (e.g., [12, 13, 21])

The spaces of independent and dependent variables will be \mathbb{R}^p and \mathbb{R}^q respectively, or their subsets Ω and U (we set $M = \Omega \times U$). By $(x, u^{(n)})$ we shall denote a point in the n -th order prolongation or jet space $M^{(n)}$, whose components are x^i , $i = 1, \dots, p$, of x and u^α_j , $\alpha = 1, \dots, q$, $J = (j_1, \dots, j_k)$, $1 \leq j_k \leq p$, $1 \leq k \leq n$, representing u and all partial derivatives of u up to order n . The n -th prolongation of a function $f : \Omega \rightarrow U$ will be the function $\text{pr}^{(n)}f : \Omega \rightarrow U^{(n)}$ with components representing f and its partial derivatives up to order n . In general, the n -th prolongation of any object will be denoted by $\text{pr}^{(n)}$. By G we shall denote a one-parameter Lie group of transformations which acts on M and write for the action $(x', u') = g_\eta \cdot (x, u) = \Phi(\eta, (x, u)) = (\Xi(\eta, (x, u)), \Psi(\eta(x, u)))$. To simplify notation we will also write $\Phi_\eta(x, u)$, $\Xi_\eta(x, u)$ and $\Psi_\eta(x, u)$. G is called projectable if $\Xi_\eta(x, u) = \Xi_\eta(x)$. The infinitesimal generator of G is a vector field on M denoted by \mathbf{v} and written in the form $\sum_{i=1}^p \xi^i(x, u) \partial_{x^i} + \sum_{\alpha=1}^q \phi_\alpha(x, u) \partial_{u^\alpha}$. The total derivative with respect to x^i will be denoted by D_i .

The algebras of generalized functions on which our approach is based will be those of special (or simplified) type ([16, Ch. 1]), so we will henceforth drop the adjective “special” and simply refer to them as Colombeau algebras. The algebra $\mathcal{G}(\Omega)$ is the quotient $\mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where

$$\begin{aligned} \mathcal{E}_M(\Omega) &= \{(u_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty(\Omega))^{(0,1]} \mid \forall K \subset\subset \Omega \ \forall \alpha \in \mathbb{N}_0^p \ \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \rightarrow 0\} \\ \mathcal{N}(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} \mid \forall K \subset\subset \Omega \ \forall \alpha \in \mathbb{N}_0^p \ \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^m) \text{ as } \varepsilon \rightarrow 0\}. \end{aligned}$$

$\mathcal{N}(\Omega)$ is an ideal in the algebra $\mathcal{E}_M(\Omega)$ (all operations are defined componentwise, i.e., for fixed ε). Elements of $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are called moderate, resp. negligible nets of smooth functions. $\mathcal{G}(\Omega)$ is an associative, commutative, differential algebra whose elements are equivalence classes $u = [(u_\varepsilon)_\varepsilon]$. $\Omega \rightarrow \mathcal{G}(\Omega)$ is a fine sheaf of differential algebras on \mathbb{R}^p . In particular, there is a

well-defined notion of support in $\mathcal{G}(\Omega)$ and we denote by $\mathcal{G}_c(\Omega)$ the space of compactly supported generalized functions in $\mathcal{G}(\Omega)$. Moreover, $\mathcal{G}(\Omega)$ contains the space $\mathcal{D}'(\Omega)$ as a linear subspace and $\mathcal{C}^\infty(\Omega)$ is a faithful subalgebra: embedding is effected basically by convolution with a fixed mollifier. This embedding is a sheaf morphism that commutes with partial differentiation. The concept of association provides a means of assigning macroscopic properties to elements of Colombeau algebras: elements u, v of a Colombeau algebra are called associated if $u_\varepsilon - v_\varepsilon \rightarrow 0$ in \mathcal{D}' . u is associated to the distribution w if $u_\varepsilon \rightarrow w$ weakly. These notions are independent of the chosen representatives.

Given another open set $\Omega' \subseteq \mathbb{R}^q$, we may consider those elements of $\mathcal{G}(\Omega)^q$ possessing a representative $(u_\varepsilon)_\varepsilon$ such that $u_\varepsilon(\Omega) \subseteq \Omega'$ for all ε and which is compactly bounded or c -bounded in the sense that

$$\forall K \subset\subset \Omega \exists K' \subset\subset \Omega' \exists \varepsilon_0 \in (0, 1] \forall \varepsilon < \varepsilon_0 : u_\varepsilon(K) \subseteq K'.$$

The space of all c -bounded generalized functions from Ω to Ω' is denoted by $\mathcal{G}[\Omega, \Omega']$. Elements of $\mathcal{G}[\Omega, \Omega']$ can be composed unrestrictedly as well as inserted into elements of $\mathcal{G}(\Omega')$. A similar definition can be given for smooth manifolds instead of open sets and a functorial theory of generalized functions based on this notion has been developed in [25, 30].

The algebra of tempered generalized functions $\mathcal{G}_\tau(\Omega)$ is defined as the quotient $\mathcal{E}_\tau(\Omega)/\mathcal{N}_\tau(\Omega)$, where

$$\begin{aligned} \mathcal{E}_\tau(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} | \forall \alpha \in \mathbb{N}_0^p \exists N \in \mathbb{N} : \sup_{x \in \Omega} (1 + |x|)^{-N} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \rightarrow 0\} \\ \mathcal{N}_\tau(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} | \forall \alpha \in \mathbb{N}_0^p \exists l \in \mathbb{N} \forall m \in \mathbb{N} : \sup_{x \in \Omega} (1 + |x|)^{-l} |u_\varepsilon(x)| = O(\varepsilon^m) \text{ as } \varepsilon \rightarrow 0\} \end{aligned}$$

The algebra \mathcal{G}_τ can be used to implement Fourier transformation in the Colombeau setting (analogous to the space \mathcal{S}' in distribution theory, which in turn is embedded into \mathcal{G}_τ). Our main interest in it, however, relies on the fact that insertion of elements of \mathcal{G} into elements of \mathcal{G}_τ yields well-defined elements of \mathcal{G} .

We shall also make use of the following 'mixed' variant of Colombeau algebra: let $\tilde{\mathcal{G}}_\tau(\Omega \times \Omega') = \tilde{\mathcal{E}}_\tau(\Omega \times \Omega')/\tilde{\mathcal{N}}_\tau(\Omega \times \Omega')$, where Ω' is a subset of $\mathbb{R}^{p'}$, and

$$\begin{aligned} \tilde{\mathcal{E}}_\tau(\Omega \times \Omega') &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega \times \Omega')^{(0,1]} | \forall K \subset\subset \Omega \forall \alpha \in \mathbb{N}_0^{p+p'} \exists N \in \mathbb{N} : \\ &\quad \sup_{x \in K, y \in \Omega'} (1 + |y|)^{-N} |\partial^\alpha u_\varepsilon(x, y)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \rightarrow 0\} \\ \tilde{\mathcal{N}}_\tau(\Omega \times \Omega') &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega \times \Omega')^{(0,1]} | \forall K \subset\subset \Omega \forall \alpha \in \mathbb{N}_0^{p+p'} \exists l \in \mathbb{N} \forall m \in \mathbb{N} : \\ &\quad \sup_{x \in K, y \in \Omega'} (1 + |y|)^{-l} |\partial^\alpha u_\varepsilon(x, y)| = O(\varepsilon^m) \text{ as } \varepsilon \rightarrow 0\}. \end{aligned}$$

Thus, its elements satisfy \mathcal{G} -estimates in the Ω -variable and \mathcal{G}_τ -estimates in the Ω' -variable.

It may be seen as a nonstandard feature of Colombeau algebras that, due to the presence of infinitesimals, their elements are not uniquely determined by their values on the usual points in their respective domains. In order to achieve such a unique determination one has to use so-called generalized points in the following sense ([39]): The space of generalized points $\tilde{\Omega}$ is the set of all equivalence classes in

$$\Omega_M = \{(x_\varepsilon)_\varepsilon \in \Omega^{(0,1]} | \exists N \in \mathbb{N} : |x_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \rightarrow 0\},$$

where the equivalence relation is defined by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall m \in \mathbb{N} : |x_\varepsilon - y_\varepsilon| = O(\varepsilon^m) \text{ as } \varepsilon \rightarrow 0,$$

i.e. $\tilde{\Omega} = \Omega_M / \sim$. The set of compactly supported points is

$$\tilde{\Omega}_c = \{\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega} | \exists K \subset\subset \Omega \exists \varepsilon_0 \in (0, 1] \forall \varepsilon < \varepsilon_0 : x_\varepsilon \in K\}.$$

If $\Omega = \mathbb{R}^p$ we shall write $\tilde{\mathbb{R}}^p$ resp. $\tilde{\mathbb{R}}_c^p$. As a special case, when $\Omega = \mathbb{R}$, we obtain $\tilde{\mathbb{R}}$, the ring of constants in any of the above algebras. Elements of $\mathcal{G}(\Omega)$ are uniquely determined by their

point values in all compactly supported generalized points in Ω . If Ω is an n -dimensional box and $u \in \mathcal{G}_\tau(\Omega)$ then $u = 0$ in $\mathcal{G}_\tau(\Omega)$ if and only if $u(\tilde{x}) = 0$ in $\tilde{\mathbb{R}}$, for all $\tilde{x} \in \tilde{\Omega}$. Moreover, one can show that Colombeau generalized functions are in fact uniquely determined by their values in all near-standard points. Here, a point $\tilde{x} \in \tilde{\Omega}_c$ is called near-standard if there exists $x \in \Omega$ such that $x_\varepsilon \rightarrow x$ as $\varepsilon \rightarrow 0$ for any representative $(x_\varepsilon)_\varepsilon$ of \tilde{x} .

The above notions can be employed to define generalized Lie group actions in the Colombeau setting (cf. [26, 8, 23]): A generalized group action on \mathbb{R}^p is an element $\Phi \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^p)^p$ which is a one-parameter group in the sense that $\Phi(0, \cdot) = \text{id}$ in $\mathcal{G}_\tau(\mathbb{R}^p)^p$ and $\Phi(\eta_1 + \eta_2, \cdot) = \Phi(\eta_1, \Phi(\eta_2, \cdot))$ in $\tilde{\mathcal{G}}_\tau(\mathbb{R}^2 \times \mathbb{R}^p)^p$. If $\xi \in \mathcal{G}_\tau(\mathbb{R}^p)^p$ is a generalized vector field with the property that there is a unique generalized group action $\Phi \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^p)^p$ satisfying $\frac{d}{d\eta}\Phi(\eta, x) = \xi(\Phi(\eta, x))$ in $\tilde{\mathcal{G}}_\tau(\mathbb{R}^{1+p})^p$ (i.e., if Φ is the unique flow of ξ) then ξ is called the infinitesimal generator of Φ and both ξ and Φ are called \mathcal{G} -complete. If, in addition, Φ and ξ have representatives $(\Phi_\varepsilon)_\varepsilon$ and $(\xi_\varepsilon)_\varepsilon$ such that Φ_ε is the flow of ξ_ε for each $\varepsilon \in (0, 1]$, then Φ and ξ are called strictly \mathcal{G} -complete.

The remainder of this introduction will be devoted to the notion of positivity in the generalized functions setting. Since the results in this area are rather recent and come from a variety of different sources, we collect below those parts of the theory which will be crucial for our further investigation (cf. [16, 40, 32]).

By [16, Theorem 1.2.38], an element \tilde{x} of $\tilde{\mathbb{R}}$ is invertible if and only if it is *strictly nonzero* in the following sense: for any representative $(x_\varepsilon)_\varepsilon$ of \tilde{x} there exist $a > 0$ and ε_0 such that $|x_\varepsilon| \geq \varepsilon^a$, for all $\varepsilon < \varepsilon_0$. This, in turn, is equivalent to \tilde{x} not being a zero divisor ([16, Theorem 1.2.39]).

$\tilde{x} \in \tilde{\mathbb{R}}$ is said to be nonnegative, $\tilde{x} \geq 0$, if there exists a representative such that each x_ε is nonnegative. Equivalently, for each representative $(x_\varepsilon)_\varepsilon$ of \tilde{x} and each $a > 0$ there exists ε_0 such that $x_\varepsilon + \varepsilon^a \geq 0$, for all $\varepsilon < \varepsilon_0$. $\tilde{x} \in \tilde{\mathbb{R}}$ is said to be strictly positive, $\tilde{x} > 0$, if it is positive and invertible. By the above, this means that there exists $a > 0$ such that for each representative $(x_\varepsilon)_\varepsilon$ of \tilde{x} there exists ε_0 with $x_\varepsilon > \varepsilon^a$, for all $\varepsilon < \varepsilon_0$. Note that since $\tilde{\mathbb{R}}$ contains zero divisors, $\tilde{x} > 0$ is strictly stronger than $\tilde{x} \geq 0$ and $\tilde{x} \neq 0$. We shall need the following characterizations from [32]: $\tilde{x} \in \tilde{\mathbb{R}}$ is strictly positive if and only if

- (i) \tilde{x} is invertible and has a representative $(x_\varepsilon)_\varepsilon$ such that $x_\varepsilon > 0, \forall \varepsilon \in (0, 1]$;
- (ii) for each representative $(x_\varepsilon)_\varepsilon$ of \tilde{x} there exists an ε_0 such that $x_\varepsilon > 0, \forall \varepsilon < \varepsilon_0$.

As a corollary we have: if \tilde{x} is a near-standard point associated with $x \in \mathbb{R}$ then

- (i) if $x \neq 0$ (resp. $x > 0$) then \tilde{x} is invertible (resp. strictly positive);
- (ii) if $\tilde{x} \geq 0$ then $x \geq 0$.

Note that the converse implications do not hold as demonstrated by the following simple examples: take $\tilde{x} := [(\varepsilon)_\varepsilon]$. Then \tilde{x} is invertible (moreover strictly positive), but $\tilde{x} \approx 0$. Also, $[(-\varepsilon)_\varepsilon] \approx 0$, but $[(-\varepsilon)_\varepsilon] < 0$. We shall repeatedly make use of the following result:

Lemma 1.1 *If $\tilde{x}, \tilde{y} \in \tilde{\mathbb{R}}$ and $|\tilde{x}| \leq \tilde{s} \cdot |\tilde{y}|$, for all invertible $\tilde{s} \in \tilde{\mathbb{R}}$ with $0 < \tilde{s} \leq s_0$ for some $0 < s_0 \in \mathbb{R}$, then $\tilde{x} = 0$ in $\tilde{\mathbb{R}}$.*

Proof. Suppose $\tilde{x} \neq 0$ in $\tilde{\mathbb{R}}$. Then for any representative $(x_\varepsilon)_\varepsilon$ of \tilde{x} there is $l \in \mathbb{R}$ and a sequence $\varepsilon_k \searrow 0$ such that

$$|x_{\varepsilon_k}| > \varepsilon_k^l, \quad \forall k \in \mathbb{N}.$$

Further, by assumption, for any representatives $(y_\varepsilon)_\varepsilon$ of \tilde{y} and $(s_\varepsilon)_\varepsilon$ of \tilde{s} and for all $a > 0$

$$\exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 : \quad s_\varepsilon \cdot |y_\varepsilon| - |x_\varepsilon| + \varepsilon^a \geq 0.$$

In particular, this inequality holds for each ε_k and $k \in \mathbb{N}$ large enough. Fix $(y_\varepsilon)_\varepsilon$ and choose $N \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ for which $|y_{\varepsilon_k}| \leq \varepsilon_k^{-N}, \forall k > k_0$. Then

$$s_{\varepsilon_k} \cdot \varepsilon_k^{-N} > \varepsilon_k^l - \varepsilon_k^a > \varepsilon_k^{l+1},$$

for a, k large enough, hence

$$s_{\varepsilon_k} > \varepsilon_k^{N+l+1}.$$

Since this inequality would have to hold for all invertible $\tilde{s} \in \tilde{\mathbb{R}}$, $0 < \tilde{s} \leq s_0$, we arrive at a contradiction (e.g. take $\tilde{s} = [(\varepsilon^{N+l+2})_\varepsilon]$). \square

Remark 1.2 The above proof in fact shows that it suffices to check the assumption for all $\tilde{s} = [(\varepsilon^m)_\varepsilon]$ with $m > 0$.

Turning now to non-constant generalized functions we recall from [40, 32] the definitions of non-negative and strictly positive generalized functions as well as a characterization of positivity: $f \in \mathcal{G}(\Omega)$ ($\Omega \subseteq \mathbb{R}^p$ open) is said to be nonnegative, $f \geq 0$, if $\forall (f_\varepsilon)_\varepsilon \forall K \subset\subset \Omega \forall a > 0 \exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 : \inf_{x \in K} f_\varepsilon(x) + \varepsilon^a \geq 0$. f is said to be strictly positive, $f > 0$, if $\forall (f_\varepsilon)_\varepsilon \forall K \subset\subset \Omega \exists a > 0 \exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 : \inf_{x \in K} f_\varepsilon(x) > \varepsilon^a$. The latter can be characterized as follows: a generalized function $f \in \mathcal{G}(\Omega)$ is strictly positive if and only if $\forall (f_\varepsilon)_\varepsilon \forall K \subset\subset \Omega \exists \varepsilon_0 \forall x \in K \forall \varepsilon < \varepsilon_0 : f_\varepsilon(x) > 0$. Moreover, by [16, Theorem 1.2.5], f is invertible in $\mathcal{G}(\Omega)$ if and only if $|f|$ is strictly positive.

We conclude this brief overview by recalling the notion of positive (semi)definiteness of symmetric bilinear forms on $\tilde{\mathbb{R}}^p$. For this purpose we first need the definition of a free generalized vector (cf. [32]): let $\tilde{\xi} \in \tilde{\mathbb{R}}^p$ and $\tilde{\xi} \neq 0$ in $\tilde{\mathbb{R}}^p$. We call $\tilde{\xi}$ free if whenever $\tilde{\lambda} \in \tilde{\mathbb{R}}$ and $\tilde{\lambda} \cdot \tilde{\xi} = 0$ then it follows that $\tilde{\lambda} = 0$ in $\tilde{\mathbb{R}}$. Let $A \in \tilde{\mathbb{R}}^{p^2}$ be a symmetric matrix (i.e. $A = A^\top$ in $\tilde{\mathbb{R}}^{p^2}$). A is positive semidefinite if for any $\tilde{\xi} \in \tilde{\mathbb{R}}^p$, $\tilde{\xi}^\top A \tilde{\xi} \geq 0$ in $\tilde{\mathbb{R}}$. A is positive definite if for all free vectors $\tilde{\xi} \in \tilde{\mathbb{R}}^p$, $\tilde{\xi}^\top A \tilde{\xi}$ is strictly positive (i.e., invertible and positive) in $\tilde{\mathbb{R}}$. Equivalent characterizations for positive definiteness, resp. semidefiniteness of a symmetric matrix $A \in \tilde{\mathbb{R}}^{p^2}$ are:

- (i) all eigenvalues of A are strictly positive (resp. nonnegative);
- (ii) A has a representative $(A_\varepsilon)_\varepsilon$ such that each A_ε is symmetric and positive definite (resp. positive semidefinite);
- (iii) for each symmetric representative $(A_\varepsilon)_\varepsilon$ of A there exists ε_0 such that A_ε is positive definite (resp. positive semidefinite), for all $\varepsilon < \varepsilon_0$.

2 Differential calculus in algebras of generalized functions

Basic necessary and sufficient conditions for extremals in the calculus of variations are usually based on the following elementary results on real valued functions: Let f map Ω to \mathbb{R} , where Ω is an open subset of \mathbb{R}^p . Let $x_0 \in \Omega$ be a local minimum (or maximum) of $f \in \mathcal{C}^1(\Omega)$. Then all first order partial derivatives of f vanish at x_0 . Such points are called critical (or stationary) points. When $f \in \mathcal{C}^2(\Omega)$ then, in addition, the Hessian matrix of f is positive (or negative) semidefinite at x_0 . A sufficient condition for a local extremum of $f \in \mathcal{C}^2(\Omega)$ is as follows: if $x_0 \in \Omega$ is a critical point of f such that $D^2 f(x_0)$ is positive (or negative) definite on \mathbb{R}^p , then x_0 is a local minimum (or maximum) of f .

In this section we want to derive analogous criteria for extremals of generalized functions. The main difficulty in carrying out this task lies in the algebraic structure of $\tilde{\mathbb{R}}$: the existence of zero divisors necessitates certain adaptations in the classical arguments and provides for a more complex structure of the attainable results. In this paper we are interested mainly in the extrema of generalized functions in classical points (i.e. in $x \in \mathbb{R}^p$) as this suffices for the applications in the calculus of variations we aim at. We will however add remarks which give analogous criteria for extrema in generalized points.

As in the classical case, we define a local minimum (resp. maximum) of $f \in \mathcal{G}(\Omega)$ (Ω is an open subset of \mathbb{R}^p) to be an $x_0 \in \Omega$ with the property that there exists a neighborhood $\Omega' \subseteq \Omega$ of x_0 such that $f(x_0) \leq f(\tilde{x})$ (resp. $f(\tilde{x}) \leq f(x_0)$), for all $\tilde{x} \in \tilde{\Omega}'_c$.

Our first result provides necessary conditions for a minimum:

Proposition 2.1 *Let $f \in \mathcal{G}(\Omega)$. If $x_0 \in \Omega$ is a local minimum of f then*

(i) $Df(x_0) = 0$ in $\tilde{\mathbb{R}}^p$;

(ii) $D^2f(x_0) \in \tilde{\mathbb{R}}^{p^2}$ is positive semidefinite.

Proof. Without loss of generality we may suppose that $x_0 = 0$.

(i) For any $i \in \{1, \dots, p\}$, let $g_i(\cdot) := f(0, \dots, 0, \cdot, 0, \dots, 0)$ be the i -th partial function. Then $g_i \in \mathcal{G}(\Omega_i)$ ($\Omega_i = \{t \in \mathbb{R} \mid (0, \dots, t, \dots, 0) \in \Omega\}$ open in \mathbb{R}) and 0 is a local minimum of g_i on Ω_i . We have to show that $g'_i(0) = \partial_i f(0) = 0$ in $\tilde{\mathbb{R}}$ for all i . Let $i \in \{1, \dots, p\}$ and assume that 0 minimizes g_i over $(-s_0, s_0)_c$ ($s_0 > 0$). Let $\tilde{s} = [(s_\varepsilon)_\varepsilon] \in (-s_0, s_0)_c$ and take any representative $(g_{i\varepsilon})_\varepsilon$ of g_i . Then for all ε we have a Taylor expansion of $g_{i\varepsilon}$ at 0:

$$g_{i\varepsilon}(s_\varepsilon) = g_{i\varepsilon}(0) + s_\varepsilon g'_{i\varepsilon}(0) + \frac{s_\varepsilon^2}{2} g''_{i\varepsilon}(\theta_\varepsilon s_\varepsilon),$$

with $0 < \theta_\varepsilon < 1$. $g_i(0) \leq g_i(\tilde{s})$ says that

$$\forall a > 0 \exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 : g_{i\varepsilon}(s_\varepsilon) - g_{i\varepsilon}(0) + \varepsilon^a \geq 0.$$

Set $M_\varepsilon := \sup_{x \in [-s_0, s_0]} |g''_{i\varepsilon}(x)|$. Then

$$0 \leq s_\varepsilon g'_{i\varepsilon}(0) + \frac{s_\varepsilon^2}{2} g''_{i\varepsilon}(\theta_\varepsilon s_\varepsilon) + \varepsilon^a \leq s_\varepsilon g'_{i\varepsilon}(0) + \frac{s_\varepsilon^2}{2} M_\varepsilon + \varepsilon^a. \quad (1)$$

Suppose \tilde{s} is strictly positive (hence invertible). Then for any representative $(s_\varepsilon)_\varepsilon$ of \tilde{s} there exist $m \in \mathbb{N}$ and ε_1 such that $s_\varepsilon \geq \varepsilon^m$, for all $\varepsilon < \varepsilon_1$. Divide (1) by s_ε for $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$:

$$0 \leq g'_{i\varepsilon}(0) + \frac{M_\varepsilon}{2} |s_\varepsilon| + \varepsilon^{a'},$$

where $a' = a - m$. Otherwise, if $[(s_\varepsilon)_\varepsilon]$ is strictly negative we similarly obtain

$$0 \geq g'_{i\varepsilon}(0) - \frac{M_\varepsilon}{2} |s_\varepsilon| - \varepsilon^{a'}.$$

Collecting both cases yields

$$|g'_{i\varepsilon}(0)| \leq \frac{M}{2} |\tilde{s}|,$$

for all invertible generalized numbers $\tilde{s} = [(s_\varepsilon)_\varepsilon] \in (-s_0, s_0)_c$, with $M = [(M_\varepsilon)_\varepsilon]$. The claim now follows from Lemma 1.1.

(ii) Taylor expansion of f around 0 gives:

$$f_\varepsilon(x_\varepsilon) = f_\varepsilon(0) + Df_\varepsilon(0)(x_\varepsilon) + \frac{1}{2} x_\varepsilon^\top \cdot D^2 f_\varepsilon(0) \cdot x_\varepsilon + \frac{1}{3!} ((x_\varepsilon D)^3 f_\varepsilon)(\theta_\varepsilon x_\varepsilon).$$

Now for any $a > 0$, $f_\varepsilon(x_\varepsilon) - f_\varepsilon(0) + \varepsilon^a \geq 0$ for ε sufficiently small. Also by (i), $Df_\varepsilon(0)(x_\varepsilon) = O(\varepsilon^a)$, for all $a > 0$. Moderateness of $D^3 f$ yields the existence of some $N > 0$ such that $\sup_{|y| \leq 1} \|D^3 f(y)\| = O(\varepsilon^{-N})$. Thus if we suppose that $|x_\varepsilon| = \varepsilon^s$ with $s = \frac{a+N}{3}$ then we obtain that $x_\varepsilon^\top \cdot D^2 f_\varepsilon(0) \cdot x_\varepsilon + \varepsilon^{a-1} \geq 0$ for any given a if ε is sufficiently small.

$D^2 f(0)$ possesses a symmetric representative, so for each ε there exists an orthogonal matrix U_ε such that with $U = [(U_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}^{p^2}$, $U \cdot D^2 f(0) \cdot U^\top = \text{diag}(\lambda_1, \dots, \lambda_p)$ with $\lambda_1, \dots, \lambda_p$ the eigenvalues of $D^2 f(0)$ in $\tilde{\mathbb{R}}$ (cf. [32, Lemma 1.6]). Denote by \tilde{x}_{λ_i} the corresponding unit eigenvectors (the columns of U) and set $y_{i\varepsilon} := \varepsilon^s \cdot x_{\lambda_i \varepsilon}$. Then by the above, for each $a' > 0$ and ε small

$$0 \leq \varepsilon^{a'} + y_{i\varepsilon}^\top \cdot D^2 f_\varepsilon(0) \cdot y_{i\varepsilon} = \varepsilon^{a'} + \varepsilon^{2s} \lambda_{i\varepsilon},$$

implying that all λ_i are positive in $\tilde{\mathbb{R}}$. As mentioned in the introduction, this is equivalent to positive semidefiniteness of $D^2 f(0)$. \square

Remark 2.2 (*Extrema in generalized points*). In order to extend the validity of 2.1 to $\tilde{x}_0 \in \tilde{\Omega}_c$ we first need a concept of local minimum in this setting. We employ the notion of support of a generalized point $\tilde{x}_0 \in \tilde{\Omega}_c$ introduced in [9]: the support $\text{supp}(\tilde{x}_0)$ of \tilde{x}_0 is defined as the set of accumulation points of any representative $(x_{0\varepsilon})_\varepsilon$. We call \tilde{x}_0 a local minimum of $f \in \mathcal{G}(\Omega)$ if there exists a neighborhood Ω' of $\text{supp}(x_0)$ such that $f(\tilde{x}_0) \leq f(\tilde{x})$ for all $\tilde{x} \in \tilde{\Omega}'_c$. The proof of 2.1 (i) can be adapted to this situation: set $g_{i\varepsilon}(t) := f_\varepsilon(x_{0\varepsilon}^1, \dots, x_{0\varepsilon}^i + t, \dots, x_{0\varepsilon}^n)$. Then although there need not be a finite interval on which all $g_{i\varepsilon}$ are simultaneously defined, the above arguments may still be applied for ε small (using 1.2) and each i , showing that $Df(\tilde{x}_0) = 0$. Similarly we may modify the proof of 2.1 (ii).

The above result demonstrates that the typical necessary conditions for local minima are entirely analogous to those for minima of classical functions: the gradient vanishes and the Hessian matrix is positive semidefinite. However, as the following example shows one should not expect such a direct analogy with the smooth setting in the case of sufficient conditions: We construct a generalized function whose first derivative vanishes at zero and whose second derivative at zero is strictly positive but which does not have a minimum in 0.

Example 2.3 Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\text{supp } \varphi \subseteq [-2, 2]$, $\varphi|_{(-1/2, 1/2)} = x^2$ and $\varphi(-1) = \varphi(1) = -1$. Define $f_\varepsilon(\cdot) := \varphi(\frac{\cdot}{\varepsilon})$ and $f := [(f_\varepsilon)_\varepsilon]$. Clearly, $f \in \mathcal{G}(\mathbb{R})$, $f_\varepsilon(0) = 0$, $f'_\varepsilon(0) = 0$ and $f''_\varepsilon(0) = \frac{2}{\varepsilon^2}$, so $f''(0)$ is strictly positive (even $f''_\varepsilon(0) \rightarrow \infty$). But zero is not a local minimum since the value of f in the generalized point $[(\varepsilon)_\varepsilon]$ is -1 .

We therefore have to alter the condition of positive (semi)definiteness in zero to also take into account neighboring generalized points. Of course we would like to include as few points as possible. The following example shows that merely considering near-standard points associated to 0 is insufficient:

Example 2.4 Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\text{supp } \varphi \subseteq (-1, 1)$ and $\varphi|_{[-1/2, 1/2]} = -1$. Define

$$F_\varepsilon(x) := \sum_{n=1}^{\infty} \varepsilon^n \varphi\left(\frac{x - 1/n}{\varepsilon}\right), \quad \varepsilon \in (0, 1],$$

and $F := [(F_\varepsilon)_\varepsilon]$. Then $F \in \mathcal{G}(\mathbb{R})$, $F(0) = 0$ and $F'(0) = 0$ in $\tilde{\mathbb{R}}$. Moreover, $F''(\tilde{x}) \geq 0$ in $\tilde{\mathbb{R}}$, for all near-standard points $\tilde{x} \approx 0$ (in fact $F''(\tilde{x}) = 0$ in $\tilde{\mathbb{R}}$ for all $\tilde{x} \approx 0$). But F does not attain a local minimum at zero. For any $n_0 \in \mathbb{N}$, $F_\varepsilon(\frac{1}{n_0}) = -\varepsilon^{n_0}$ for $2\varepsilon < 1/n_0 - 1/(n_0 + 1)$, hence $F(\frac{1}{n_0}) < 0$ in $\tilde{\mathbb{R}}$. Therefore, 0 is not a local minimum of f .

As suggested by the previous examples we therefore give a sufficient condition for a local minimum which is based on the behavior of f in a neighborhood of the critical point.

Proposition 2.5 *Let $f \in \mathcal{G}(\Omega)$, $x_0 \in \Omega$, $Df(x_0) = 0$ and let $D^2f(\tilde{x})$ be positive semidefinite for all $\tilde{x} \in \tilde{\Omega}'_c$ (with Ω' a star-shaped neighborhood of x_0 in Ω). Then x_0 is a minimum of f in $\tilde{\Omega}'_c$. If in addition $D^2f(\tilde{x})$ is positive definite for all $\tilde{x} \in \tilde{\Omega}'_c$, then the minimum is unique on $\tilde{\Omega}'_c$.*

Proof. Again, taking $x_0 = 0$ is no restriction. Let $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega}'_c$. Take a representative $(f_\varepsilon)_\varepsilon$ of f . For each ε we have

$$f_\varepsilon(x_\varepsilon) = f_\varepsilon(0) + Df_\varepsilon(0)(x_\varepsilon) + \frac{1}{2}((x_\varepsilon D)^2 f_\varepsilon)(\theta_\varepsilon x_\varepsilon)$$

with $\theta_\varepsilon \in (0, 1)$. Since $Df(0) = 0$ in $\tilde{\mathbb{R}}$ it follows that for all $m > 0$ and ε small, $|Df_\varepsilon(0)(x_\varepsilon)| \leq \varepsilon^m$. On the other hand, by the assumption that $D^2f|_{\tilde{\Omega}'_c}$ is positive semidefinite we know that $\forall a' > 0 \exists \varepsilon_0$ such that $1/2((x_\varepsilon D)^2 f_\varepsilon)(\theta_\varepsilon x_\varepsilon) + \varepsilon^{a'} \geq 0, \forall \varepsilon < \varepsilon_0$. Let $a > 0$ be given and set $a' = a$ and

m to be $a + 1$. Then

$$\begin{aligned}
0 &\leq \varepsilon^{a+1} + f_\varepsilon(x_\varepsilon) - f_\varepsilon(0) - Df_\varepsilon(0)(x_\varepsilon) \\
&\leq \varepsilon^{a+1} + f_\varepsilon(x_\varepsilon) - f_\varepsilon(0) + |Df_\varepsilon(0)(x_\varepsilon)| \\
&\leq \varepsilon^{a+1} + f_\varepsilon(x_\varepsilon) - f_\varepsilon(0) + \varepsilon^{a+1} \\
&\leq f_\varepsilon(x_\varepsilon) - f_\varepsilon(0) + \varepsilon^a,
\end{aligned}$$

when $\varepsilon \rightarrow 0$. Hence $f(\tilde{x}) - f(0) \geq 0$ and the first part of lemma is proved.

To show uniqueness, let $\tilde{x} \in \tilde{\Omega}'_c$ and $\tilde{x} \neq 0$ in \mathbb{R}^p . Then there exist l and $\varepsilon_k \searrow 0$ such that $|x_{\varepsilon_k}| > \varepsilon_k^l$, for all $k \in \mathbb{N}$. Choose a representative $(f_\varepsilon)_\varepsilon$ of f with $Df_\varepsilon(0) = 0$ for all ε and write $f_\varepsilon(x_\varepsilon) = f_\varepsilon(0) + \frac{1}{2}((x_\varepsilon D)^2 f_\varepsilon)(\theta_\varepsilon x_\varepsilon)$. In order to bring to bear the positive definiteness of $D^2 f$ we embed the sequence $(x_{\varepsilon_k})_k$ into a free vector in $\tilde{\Omega}'_c$ as follows: Let \tilde{y} be any free vector in $\tilde{\Omega}'_c$ (e.g., $\tilde{y} = [(\varepsilon)_\varepsilon]$) and define \tilde{x}' as $[(x'_\varepsilon)_\varepsilon]$, where

$$x'_\varepsilon := \begin{cases} x_{\varepsilon_k}, & \varepsilon = \varepsilon_k \\ y_\varepsilon, & \text{otherwise} \end{cases}$$

Then \tilde{x}' is free, so the positive definiteness of $D^2 f$ in $\tilde{z} := [(\theta_\varepsilon x_\varepsilon)_\varepsilon] \in \tilde{\Omega}'_c$ implies strict positivity of $(\tilde{x}')^\top \cdot D^2 f(\tilde{z}) \cdot \tilde{x}'$. Thus there exists some $m > 0$ such that for large enough $k \in \mathbb{N}$

$$f_{\varepsilon_k}(x_{\varepsilon_k}) - f_{\varepsilon_k}(0) = \frac{1}{2}((x_{\varepsilon_k} D)^2 f_{\varepsilon_k})(\theta_{\varepsilon_k} x_{\varepsilon_k}) > \varepsilon_k^m.$$

This implies that $f(\tilde{x}) \neq f(0)$ in $\tilde{\mathbb{R}}$, for all $\tilde{x} \in \tilde{\Omega}'_c$, and uniqueness follows. \square

Remark 2.6 As in 2.2 the above result may be extended to generalized points \tilde{x}_0 , if we suppose Ω' to be a convex open neighborhood of $\text{supp}(\tilde{x}_0)$.

The uniqueness condition in the previous result raises the question whether positive definiteness of the second derivative could be replaced by the assumption of $D^2 f$ being positive semidefinite but different from zero in $\tilde{\mathbb{R}}$. These two notions are different in $\tilde{\mathbb{R}}$, due to the existence of zero divisors. The following example, however, shows that the latter condition is too weak.

Example 2.7 For any zero divisor $\alpha \geq 0$ in $\tilde{\mathbb{R}}$ the assignment

$$f(x) := \alpha x^2,$$

defines an element of $\mathcal{G}(\mathbb{R})$ which attains its (even global) minimum 0 in the point 0. Moreover, $f'(0) = 0$ in $\tilde{\mathbb{R}}$ and $f'' = 2\alpha$ is positive and different from zero everywhere in $\tilde{\mathbb{R}}$. Nevertheless the minimum at 0 is not unique: take any $\tilde{s} \neq 0$ in $\tilde{\mathbb{R}}_c$ with $\alpha \cdot \tilde{s} = 0$, then also $f(\tilde{s}) = 0$.

We conclude this section with a characterization of nonnegativity, resp. strict positivity, of generalized functions by their values in all near-standard points. As a consequence we note that the respective assumptions in the previous results can equivalently formulated in terms of this proper subset of generalized points.

Lemma 2.8 *Let $f \in \mathcal{G}(\Omega)$. Then $f \geq 0$ in $\mathcal{G}(\Omega)$ (resp. $f > 0$ in $\mathcal{G}(\Omega)$) if and only if $f(\tilde{x}) \geq 0$ in $\tilde{\mathbb{R}}$ (resp. $f(\tilde{x}) > 0$ in $\tilde{\mathbb{R}}$) for all near-standard points $\tilde{x} \in \tilde{\Omega}_c$.*

Proof. We give a proof for the characterization of nonnegativity of f , the case of strict positivity can be established along the same lines. The condition is clearly necessary. Conversely, suppose that f is not nonnegative in $\mathcal{G}(\Omega)$. Then

$$\exists (f_\varepsilon)_\varepsilon \exists a > 0 \exists K \subset \subset \Omega \forall k \in \mathbb{N} \exists \varepsilon_k < \frac{1}{k} \exists x_k \in K : f_{\varepsilon_k}(x_k) + \varepsilon_k^a < 0.$$

Without loss of generality we may assume that $\varepsilon_{k+1} < \varepsilon_k$ and that the sequence $\{x_k\}_{k \in \mathbb{N}}$ converges to some $x_0 \in K$. Set $x_\varepsilon := x_k$ for $\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$ and $\tilde{x} := [(x_\varepsilon)_\varepsilon]$. Then \tilde{x} is a near-standard point and $f_{\varepsilon_k}(x_{\varepsilon_k}) + \varepsilon_k^a < 0 \ \forall k \in \mathbb{N}$, hence $f(\tilde{x}) \not\geq 0$, a contradiction. \square

Inspection of the above proof shows that in fact the subset of so-called sequential near-standard points suffices to characterize nonnegativity resp. positivity of generalized functions. Here, a near-standard point $\tilde{x} \approx x$ is called sequential if it has a representative $(x_\varepsilon)_\varepsilon$ of the following form: $\exists \varepsilon_k \searrow 0 \ \exists x_{\varepsilon_k}, x_{\varepsilon_k} \rightarrow x$, such that $x_\varepsilon = x_{\varepsilon_k}, \forall \varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$.

3 First and second variation

Let Ω be an open subset of \mathbb{R}^n and let $D = \Omega$ or $D = \bar{\Omega}$. Let \mathcal{A} be a subset of $\mathcal{G}(D)$ and let $\mathcal{L} : \mathcal{A} \rightarrow \tilde{\mathbb{R}}$. We want to determine extremals of \mathcal{L} under certain admissible variations. To make this problem accessible to an analytical treatment we make the following assumptions (which will automatically be satisfied in the setting most relevant to us, namely (3) below). Let \mathcal{A}_0 , the space of *admissible variations* be an $\tilde{\mathbb{R}}$ -submodule of $\mathcal{G}(D)$ such that $\mathcal{A} + \mathcal{A}_0 \subseteq \mathcal{A}$. Moreover, we suppose that for all $u \in \mathcal{A}$ and all $v \in \mathcal{A}_0$, the mapping

$$\mathcal{L}^{u,v} : s \mapsto \mathcal{L}(u + sv)$$

is an element of $\mathcal{G}(J^{u,v})$, where $J^{u,v}$ is some open interval around 0 in \mathbb{R} .

We call $u \in \mathcal{A}$ a minimizer of the functional \mathcal{L} in \mathcal{A} if $\mathcal{L}(u) \leq \mathcal{L}(w)$ in $\tilde{\mathbb{R}}$ for all $w \in \mathcal{A}$. Finally, we define the first and second variation of \mathcal{L} at $u \in \mathcal{A}$ in the direction $v \in \mathcal{A}_0$ as

$$\delta \mathcal{L}(u; v) := \left. \frac{d}{ds} \right|_{s=0} \mathcal{L}^{u,v}(s); \quad \delta^2 \mathcal{L}(u; v) := \left. \frac{d^2}{ds^2} \right|_{s=0} \mathcal{L}^{u,v}(s) \quad (2)$$

Under these assumptions we have

Proposition 3.1 *Let $u \in \mathcal{A}$ be a minimizer of the functional \mathcal{L} in \mathcal{A} . Then*

- (i) $\delta \mathcal{L}(u; v) = 0$ in $\tilde{\mathbb{R}}$ for all $v \in \mathcal{A}_0$;
- (ii) $\delta^2 \mathcal{L}(u; v) \geq 0$ in $\tilde{\mathbb{R}}$ for all $v \in \mathcal{A}_0$.

Proof. This is immediate from Proposition 2.1. \square

Turning now to our main object of study, let Ω be an open, connected and bounded subset of \mathbb{R}^p with smooth boundary $\partial\Omega$. Let $\mathcal{L} : \mathcal{G}(\bar{\Omega})^q \rightarrow \tilde{\mathbb{R}}$ be a functional of the form

$$\mathcal{L}(u) := \int_{\Omega} L(x, u^{(n)}(x)) \, dx, \quad (3)$$

where $L \in \mathcal{G}_\tau((\bar{\Omega} \times \mathbb{R}^q)^{(n)})$. We will write \mathcal{L}_ε for the functional $u \mapsto \int_{\Omega} L_\varepsilon(x, u) \, dx$, where $u \in \mathcal{C}^\infty(\bar{\Omega})$ and $(L_\varepsilon)_\varepsilon$ is any representative of L .

We seek a minimizer u among all functions in $\mathcal{G}(\bar{\Omega})^q$ which additionally satisfy the boundary condition $u|_{\partial\Omega} = u_0$, $u_0 \in \mathcal{G}(\partial\Omega)^q$ (here $\partial\Omega$ is viewed as a $(p-1)$ -dimensional submanifold of \mathbb{R}^p). The integrand L is called the Lagrangian of the variational problem. In the notation introduced above we have

$$\mathcal{A} = \{w \in \mathcal{G}(\bar{\Omega})^q \mid w|_{\partial\Omega} = u_0\} \quad (4)$$

for some $u_0 \in \mathcal{G}(\partial\Omega)^q$, with the corresponding space of admissible variations given by

$$\mathcal{A}_0 = \{w \in \mathcal{G}(\bar{\Omega})^q \mid w|_{\partial\Omega} = 0\}. \quad (5)$$

Note that the integral in (3) is well-defined since the composition of a tempered generalized function with an element of \mathcal{G} is a well-defined element of \mathcal{G} (cf. [16, Proposition 1.2.30]). Also, (3) is the most general form of the variational problem, involving p independent, q dependent variables and all partial derivatives of dependent variables up to order n . In some cases, in order to simplify notation we are going to consider only the case of one unknown function; the generalization to two or more dimensions is then a technical matter.

Remark 3.2 In the formulation of the variational problem, instead of tempered generalized functions one may alternatively use c -bounded generalized functions: assume that $L \in \mathcal{G}((\bar{\Omega} \times \mathbb{R}^q)^{(n)})$. Since c -bounded generalized functions may be composed unrestrictedly (see [16, Proposition 1.2.8]), a minimizer then can be sought among all c -bounded generalized functions with prescribed boundary conditions. In fact, for the integrand to be well-defined not only u but also all partial derivatives of u up to order n have to be c -bounded, i.e. the admissible set is $\mathcal{A} := \{w | w^{(n)} \in \mathcal{G}[\bar{\Omega}, (\mathbb{R}^q)^{(n)}] \wedge w|_{\partial\Omega} = u_0\}$. Although convenient for an invariant geometric formulation of the theory, this latter requirement necessitates certain unwanted restrictions on the degree of singularity of u (e.g., δ -like singularities are excluded). We will therefore mainly operate in the tempered setting, but will simultaneously keep track of the c -bounded case.

In order to derive Euler-Lagrange equations in our setting, as in the classical case we first derive a fundamental lemma of the calculus of variations:

Lemma 3.3 (*Fundamental lemma of the calculus of variations*) *Let $u \in \mathcal{G}(\Omega)$. If*

$$\int_{\Omega} u(x) \varphi(x) dx = 0 \quad \text{in } \tilde{\mathbb{R}}$$

for all $\varphi \in \mathcal{G}_c(\Omega)$ then $u \equiv 0$ in $\mathcal{G}(\Omega)$.

Proof. Suppose to the contrary that $u = [(u_{\varepsilon})_{\varepsilon}]$ with $(u_{\varepsilon})_{\varepsilon} \notin \mathcal{N}(\Omega)$. Then $\exists K \subset \subset \Omega \exists l > 0 \forall k \in \mathbb{N} \exists \varepsilon_k \searrow 0 \exists x_k \in K$ such that $|u_{\varepsilon_k}(x_k)| > \varepsilon_k^l$. On the other hand, $(u_{\varepsilon})_{\varepsilon}$ is moderate, so $\exists N > 0$ such that $\sup_{x \in K} |Du_{\varepsilon}(x)| \leq \varepsilon^{-N}$. We may choose some k_0 such that for $k \geq k_0$ and any $x \in \Omega$ with $|x - x_k| \leq \frac{1}{2}\varepsilon_k^{l+N}$, the entire line segment connecting x and x_k is contained in Ω . Then

$$u_{\varepsilon_k}(x) = u_{\varepsilon_k}(x_k) + Du_{\varepsilon_k}(x_k + \theta_k(x - x_k))(x - x_k),$$

for some $\theta_k \in (0, 1)$, so that $|u_{\varepsilon_k}(x)| \geq \varepsilon_k^l - \varepsilon_k^{-N}|x - x_k| \geq \frac{1}{2}\varepsilon_k^l$ for all x with $|x - x_k| \leq \frac{1}{2}\varepsilon_k^{l+N}$. Choose $\varphi \in \mathcal{C}_0^\infty(B_{\frac{1}{2}}(0))$, $\varphi \geq 0$ and $\varphi \equiv 1$ on $B_{\frac{1}{4}}(0)$ ($B_r(0)$ denoting the open ball of radius r around 0). Define $\varphi_{\varepsilon} \in \mathcal{D}(\Omega)$ by

$$\varphi_{\varepsilon}(x) := \varepsilon^{-l-N} \varphi(\varepsilon^{-l-N}(x - x_{\varepsilon})) \cdot \text{sign}(u_{\varepsilon}(x_{\varepsilon}))$$

where $x_{\varepsilon} = x_{\varepsilon_k}$ for all $\varepsilon \in (\varepsilon_{k+1}, \varepsilon_k]$ and all $k \geq k_0$. Then $\varphi := [(\varphi_{\varepsilon})_{\varepsilon}] \in \mathcal{G}_c(\Omega)$. Since u_{ε_k} does not change sign on the support of φ_{ε_k} and since $u_{\varepsilon_k}(x) \cdot \varphi_{\varepsilon_k}(x) \geq \frac{1}{2}\varepsilon_k^{-N}$ whenever $|x - x_k| \leq \frac{1}{4}\varepsilon_k^{l+N}$, we obtain

$$\begin{aligned} \left| \int_{\Omega} u_{\varepsilon_k}(y) \varphi_{\varepsilon_k}(y) dy \right| &= \int_{B_{\frac{1}{2}\varepsilon_k^{l+N}}(x_k)} u_{\varepsilon_k}(y) \varphi_{\varepsilon_k}(y) dy \\ &\geq \int_{B_{\frac{1}{4}\varepsilon_k^{l+N}}(x_k)} u_{\varepsilon_k}(y) \varphi_{\varepsilon_k}(y) dy \\ &\geq \text{const} \cdot \varepsilon_k^{p(l+N)-N}, \end{aligned}$$

which implies that $(\int_{\Omega} u_{\varepsilon}(y) \varphi_{\varepsilon}(y) dy)_{\varepsilon} \notin \mathcal{N}(\Omega)$, a contradiction. Therefore, $u \equiv 0$ in $\mathcal{G}(\Omega)$. \square

As in the smooth setting, this result allows to derive the Euler Lagrange equations as necessary conditions for a minimizer of (3):

Theorem 3.4 *Let $u \in \mathcal{A}$ be a minimizer of (3). Then u is a solution of the system of Euler-Lagrange equations*

$$E_{\alpha}(L) = 0, \quad \alpha = 1, \dots, q, \quad \text{in } \mathcal{G}(\Omega), \quad (6)$$

where E_{α} is the α -th Euler operator

$$E_{\alpha} = \sum_J (-D)_J \frac{\partial}{\partial u_J^{\alpha}}$$

(the sum extends over all multi-indices $J = (j_1, \dots, j_k), 1 \leq j_k \leq p, 1 \leq k \leq n$).

Proof. By Proposition 3.1 (i), $\delta\mathcal{L}(u; v) = 0, \forall v \in \mathcal{A}_0$. Now

$$\begin{aligned}\delta\mathcal{L}(u; v) &= \int_{\Omega} \frac{d}{ds} \Big|_{s=0} L(x, (u + sv)^{(n)}(x)) \, dx \\ &= \int_{\Omega} \sum_{\alpha=1}^q \sum_J \frac{\partial L}{\partial u_J^\alpha}(x, u^{(n)}(x)) \cdot \partial_J v^\alpha(x) \, dx.\end{aligned}$$

Integration by parts gives (with D denoting the total derivative)

$$\delta\mathcal{L}(u; v) = \int_{\Omega} \sum_{\alpha=1}^q \left[\sum_J (-D)_J \frac{\partial L}{\partial u_J^\alpha}(x, u^{(n)}(x)) \right] \cdot v^\alpha(x) \, dx$$

The claim therefore follows from Lemma 3.3. \square

Remark 3.5 In the case $n = 1$, i.e. when $\mathcal{L}(u) = \int_{\Omega} L(x, \text{pr}^{(1)}u(x)) \, dx$, the α -th Euler operator takes the well known form:

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - \sum_{i=1}^p \frac{\partial}{\partial x_i} \frac{\partial L}{\partial u_i^\alpha}$$

where $u_i^\alpha = \partial u^\alpha / \partial x_i$.

The following result provides sufficient conditions for minimizers of the variational problem (3).

Proposition 3.6 *With \mathcal{L} as in (3) and $\mathcal{A}, \mathcal{A}_0$ as in (4), (5), suppose that*

- (i) $\delta\mathcal{L}(u; v) = 0$ in $\tilde{\mathbb{R}}$ for all $v \in \mathcal{A}_0$.
- (ii) $\delta^2\mathcal{L}(w; v) \geq 0$ for all $w \in \mathcal{A}$ and $v \in \mathcal{A}_0$.

Then u is a minimizer for \mathcal{L} . If, in addition, $s \mapsto \delta^2\mathcal{L}(u + sv; v) \neq 0$ in $\mathcal{G}([0, 1])$, $\forall v \in \mathcal{A}_0, v \neq 0$, then the minimizer is unique.

Proof. Given any $w \in \mathcal{A}$, set $\Psi := s \mapsto \mathcal{L}(u + s(w - u)) \in \mathcal{G}(\mathbb{R})$. By assumption we have $\Psi'(0) = \delta\mathcal{L}(u; w - u) = 0$ (since $w - u \in \mathcal{A}_0$) and $\Psi''(\tilde{s}) = \delta^2\mathcal{L}(u + \tilde{s}(w - u); w - u) \geq 0, \forall \tilde{s} \in \tilde{\mathbb{R}}_c$. We now apply Lemma 2.5 to conclude that 0 is a minimum of Ψ , i.e. $\Psi(0) \leq \Psi(\tilde{s}), \forall \tilde{s} \in \tilde{\mathbb{R}}_c$. In particular, for $\tilde{s} = 1$ we have that $\mathcal{L}(u) \leq \mathcal{L}(w)$, and since $w \in \mathcal{A}$ was arbitrary the claim follows. If, in addition, $s \mapsto \delta^2\mathcal{L}(u + s(w - u); w - u) \neq 0$ in $\mathcal{G}([0, 1])$ for all $w \in \mathcal{A}$, i.e. $s \mapsto \Psi''(s) \neq 0$ in $\mathcal{G}([0, 1])$, then according to [17, Lemma 3.1] (which in fact is closely related to our Lemma 3.3) we obtain that

$$\Psi(1) - \Psi(0) = \int_0^1 (1 - s) \Psi''(s) \, ds \neq 0.$$

Therefore $\mathcal{L}(w) - \mathcal{L}(u) \neq 0$ for all $w \neq u$, and uniqueness follows. \square

Remark 3.7 The uniqueness criterion in Proposition 3.6 can be adapted to provide an alternative to the uniqueness condition in Proposition 2.5, to wit: under the assumptions given there, suppose additionally that $s \mapsto D^2f(x_0 + s(\tilde{y} - x_0))[\tilde{y} - x_0, \tilde{y} - x_0] \neq 0$ in $\mathcal{G}[0, 1]$ for all $\tilde{y} \in \tilde{\Omega}'_c$. Then the minimum is unique in $\tilde{\Omega}'_c$.

Example 3.8 (Quadratic forms). Suppose that for each $\varepsilon \in (0, 1]$, $a_\varepsilon : \mathcal{C}^\infty(\bar{\Omega}) \times \mathcal{C}^\infty(\bar{\Omega}) \rightarrow \mathbb{R}$ is symmetric, bilinear and continuous. We additionally suppose that $(a_\varepsilon(u_\varepsilon, v_\varepsilon))_\varepsilon \in \mathcal{E}_M$ for all $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{E}_M(\bar{\Omega})$, and $(a_\varepsilon(u_\varepsilon, v_\varepsilon))_\varepsilon \in \mathcal{N}$ for $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\bar{\Omega})$ and $(v_\varepsilon)_\varepsilon \in \mathcal{N}(\bar{\Omega})$ (where $\tilde{\mathbb{R}} = \mathcal{E}_M/\mathcal{N}$), so that

$$\begin{aligned}a : \mathcal{G}(\bar{\Omega}) \times \mathcal{G}(\bar{\Omega}) &\rightarrow \tilde{\mathbb{R}} \\ a(u, v) &:= [(a_\varepsilon(u_\varepsilon, v_\varepsilon))_\varepsilon]\end{aligned}$$

is a well-defined bilinear map. Similarly, for each $\varepsilon \in (0, 1]$ let $f_\varepsilon : \mathcal{C}^\infty(\bar{\Omega}) \rightarrow \mathbb{R}$ be linear and continuous and suppose that $(f_\varepsilon(u_\varepsilon))_\varepsilon \in \mathcal{E}_M$ (resp. \mathcal{N}) for each $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\bar{\Omega})$ (resp. $\mathcal{N}(\bar{\Omega})$). Then set $f : \mathcal{G}(\bar{\Omega}) \rightarrow \mathbb{R}$, $f(u) := [(f_\varepsilon \circ u_\varepsilon)_\varepsilon]$.

We consider the functional

$$\begin{aligned} \mathcal{L} : \mathcal{G}(\bar{\Omega}) &\rightarrow \tilde{\mathbb{R}} \\ \mathcal{L}(u) &:= \frac{1}{2}a(u, u) + f(u). \end{aligned} \quad (7)$$

Again we will write \mathcal{L}_ε for the functional $u \mapsto \frac{1}{2}a_\varepsilon(u, u) + f_\varepsilon(u)$. In this situation, condition (i) from Proposition 3.1 reads

$$\delta \mathcal{L}(u, v) = a(u, v) + f(v) = 0 \quad (8)$$

and $\delta^2 \mathcal{L}(u, v) = a(v, v)$. Thus if a is positive semidefinite in the sense that $a(u, u) \geq 0$ for all $u \in \mathcal{G}(\bar{\Omega})$ then any solution of (8) is a minimizer of \mathcal{L} .

As a concrete example let

$$\mathcal{L}(u) = \frac{1}{2} \int_{\Omega} (\alpha(x)|\nabla u(x)|^2 + \beta(x)u(x)^2) dx + \int_{\Omega} \gamma(x)u(x) dx \quad (9)$$

with $\mathcal{A}, \mathcal{A}_0$ as in (4), (5), $\alpha, \beta, \gamma \in \mathcal{G}(\bar{\Omega})$ and $\alpha \geq 0, \beta \geq 0$ on $\bar{\Omega}$. Under these assumptions, a is positive semidefinite and by the above, any solution of (8) is a minimizer of \mathcal{L} . Moreover, the minimizer is unique by Prop. 3.6 in case α is invertible on $\bar{\Omega}$. In fact, if

$$\delta^2 \mathcal{L}(u; v) = a(v, v) = \int_{\Omega} (\alpha(x)|\nabla v(x)|^2 + \beta(x)|v(x)|^2) dx = 0$$

it follows that $\nabla v \equiv 0$ in $\mathcal{G}(\Omega)$, so v is a generalized constant. If v is an element of \mathcal{A}_0 , this constant must be zero. Therefore, $\delta^2 \mathcal{L}(u, v) \neq 0$ for all $v \in \mathcal{A}_0, v \neq 0$; this in turn implies the hypotheses of Prop. 3.6.

On the other hand, uniqueness may be lost if α is not invertible: let $\Omega = (-1, 1)$, and

$$\mathcal{L}(u) = \frac{1}{2} \int_{-1}^1 \alpha |u'(x)|^2 dx + \int_{-1}^1 \alpha u(x) dx$$

with $\alpha \geq 0$ a zero divisor, say $\alpha\omega = 0$. Then $\delta^2 \mathcal{L}(u, v) = \int_{-1}^1 \alpha |v'(x)|^2 dx \geq 0$ for all v . However, both $u(x) = x^2/2 - 1/2$ and $\bar{u}(x) = \omega x^2/2 - \omega/2$ are minimizers, because they satisfy the Euler-Lagrange equation

$$\alpha(-u''(x) + 1) = 0, \quad u(-1) = u(1) = 0.$$

We now turn to an analysis of variational problems in the distributional setting. In the language of Colombeau algebras, this amounts to expressing extremal properties in terms of the concept of association:

Definition 3.9 *An element u of $\mathcal{G}(\Omega)$ is called a minimizer in the sense of association for the functional (3) if for any (hence all) representatives $(\mathcal{L}_\varepsilon)_\varepsilon$ of \mathcal{L} and $(u_\varepsilon)_\varepsilon$ of u we have*

$$\lim_{\varepsilon \rightarrow 0} (\mathcal{L}_\varepsilon(u_\varepsilon + \varphi) - \mathcal{L}_\varepsilon(u_\varepsilon)) \geq 0 \quad \forall \varphi \in \mathcal{D}(\Omega). \quad (10)$$

For a large class of quadratic forms, the following result provides a necessary and sufficient condition for a minimizer in the sense of association:

Lemma 3.10 *Let \mathcal{L} be a functional of the form (7) where a is positive semidefinite and such that $A(\varphi, \varphi) := \lim_{\varepsilon \rightarrow 0} a_\varepsilon(\varphi, \varphi)$ exists for all $\varphi \in \mathcal{D}(\Omega)$. Let $u \in \mathcal{G}(\Omega)$. Then the following are equivalent:*

- (i) u is a minimizer in the sense of association for \mathcal{L} .
- (ii) $\lim_{\varepsilon \rightarrow 0} (a_\varepsilon(u_\varepsilon, \varphi) + f_\varepsilon(\varphi)) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$.

Proof. (ii) \Rightarrow (i): Let $\varphi \in \mathcal{D}(\Omega)$. Then $\mathcal{L}_\varepsilon(u_\varepsilon + \varphi) - \mathcal{L}_\varepsilon(u_\varepsilon) = a_\varepsilon(u_\varepsilon, \varphi) + \frac{1}{2}a_\varepsilon(\varphi, \varphi) + f_\varepsilon(\varphi) \rightarrow \frac{1}{2}A(\varphi, \varphi) \geq 0$ ($\varepsilon \rightarrow 0$).

(i) \Rightarrow (ii): Again fix $\varphi \in \mathcal{D}(\Omega)$. Inserting $\tau\varphi$ instead of φ in (10), we obtain

$$\lim_{\varepsilon \rightarrow 0} (a_\varepsilon(u_\varepsilon, \varphi) + f_\varepsilon(\varphi)) + \frac{\tau}{2}A(\varphi, \varphi) \geq 0 \quad \forall \tau > 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} (a_\varepsilon(u_\varepsilon, \varphi) + f_\varepsilon(\varphi)) + \frac{\tau}{2}A(\varphi, \varphi) \leq 0 \quad \forall \tau < 0,$$

which together give the claim. \square

The following is a classical example due to Weierstrass, cf. also [15]:

Example 3.11 Let $\Omega = (-1, 1)$, $c \neq d \in \mathbb{R}$ and consider the functional

$$\mathcal{L}(u) = \int_{-1}^1 x^2 u'(x)^2 dx.$$

We want to minimize \mathcal{L} subject to the boundary conditions $u(-1) = c$, $u(1) = d$. Here, a_ε is in fact independent of ε . Denote by v the step function equal to c_1 on $(-1, 0)$ and equal to c_2 on $(0, 1)$ (v is the “expected” minimizer). Let ρ be a standard mollifier ($\rho \in \mathcal{D}(\mathbb{R})$, $\int \rho = 1$) and set $u_\varepsilon := v * \rho_\varepsilon$. Then

$$\lim_{\varepsilon \rightarrow 0} (\mathcal{L}(u_\varepsilon + \varphi) - \mathcal{L}(u_\varepsilon)) = \lim_{\varepsilon \rightarrow 0} \int_{-1}^1 (2x^2 u'_\varepsilon(x) \varphi'(x) + x^2 \varphi'(x)^2) dx \geq 0 \quad \forall \varphi \in \mathcal{D}(\Omega),$$

so $u = [(u_\varepsilon)_\varepsilon]$ is a minimizer in the sense of association. Moreover, $\mathcal{L}(u_\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$ and u is associated to v . By Lemma 3.10 (or by direct calculation), u solves the corresponding Euler-Lagrange equation in the sense of association.

Note, however, that u is not a minimizer in $\mathcal{G}(-1, 1)$. Otherwise by Theorem 3.4 it would have to satisfy the corresponding Euler-Lagrange equations (with equality in \mathcal{G}). This, however, only has the classical solution, so also in $\mathcal{G}(-1, 1)$, no solution satisfying the boundary conditions exists.

For more general variational problems of the form (3) we need additional conditions in order to have Euler-Lagrange equations (as in Lemma 3.10 (ii)) as necessary conditions for minimizers (in the sense of association):

Proposition 3.12 *For the variational problem (3), suppose that the total second order derivative of L w.r.t. the variables $u^{(n)}$ is globally bounded (independently of ε). If $u \in \mathcal{G}(\Omega)$ is a minimizer of \mathcal{L} in the sense of association, then u satisfies the Euler-Lagrange equations for \mathcal{L} in the sense of association:*

$$E_\alpha(L) \approx 0, \quad \alpha = 1, \dots, q,$$

i.e.,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} E_\alpha(L_\varepsilon)(u_\varepsilon)(x) \varphi(x) dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega) \forall \alpha.$$

Proof. To simplify notations we give the proof for $p = q = n = 1$, the general case being entirely analogous. By Taylor expansion we have

$$L_\varepsilon(x, y + h, z + k) = L_\varepsilon(x, y, z) + \partial_2 L_\varepsilon(x, y, z)h + \partial_3 L_\varepsilon(x, y, z)k + R_\varepsilon(x, y, z, h, k)$$

where

$$R_\varepsilon(x, y, z, h, k) = \left(\int_0^1 D_{y,z}^2 L_\varepsilon(x, y + th, z + tk)(1 - t) dt \right) [(h, k), (h, k)]$$

and by assumption, $|R_\varepsilon(x, y, z, h, k)| \leq C\|(h, k)\|^2$ for all ε . Let $\varphi \in \mathcal{D}(\Omega)$, $\tau \in \mathbb{R}$. Then

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0} (\mathcal{L}_\varepsilon(u_\varepsilon + \tau\varphi) - \mathcal{L}_\varepsilon(u_\varepsilon)) \\ &= \lim_{\varepsilon \rightarrow 0} \left(\tau \int_{\Omega} \partial_2 L_\varepsilon(x, u_\varepsilon(x), u'_\varepsilon(x)) \varphi(x) + \partial_3 L_\varepsilon(x, u_\varepsilon(x), u'_\varepsilon(x)) \varphi'(x) dx \right. \\ &\quad \left. + \tau^2 \left(\int_{\Omega} R_\varepsilon(x, u_\varepsilon(x), u'_\varepsilon(x), \tau\varphi(x), \tau\varphi'(x)) [(\varphi(x), \varphi'(x)), (\varphi(x), \varphi'(x))] dx \right) \right) \end{aligned}$$

Since the second integral is uniformly bounded, dividing by τ and distinguishing the cases $\tau > 0$ and $\tau < 0$ (cf. the proof of Lemma 3.10) implies the result. \square

4 Variational symmetries and Nöther's Theorem

In this section we turn our attention to symmetries of variational problems involving generalized functions. After a quick recapitulation of the relevant notations and definitions in the smooth setting (following [42, Ch. 4]), our goal is to introduce an appropriate notion of generalized variational symmetry groups, retaining the fundamental properties of classical variational symmetries. Based on this, we will then derive a corresponding version of Nöther's theorem.

Turning first to the smooth setting, let $\mathcal{L} : \mathcal{C}^\infty(\bar{\Omega}) \rightarrow \mathbb{R}$ be a functional of the form

$$\mathcal{L}(u) = \int_{\Omega} L(x, u^{(n)}(x)) dx, \quad (11)$$

where Ω is an open, connected set in \mathbb{R}^p with smooth boundary $\partial\Omega$ and L is a smooth function on the n -jet space $(\bar{\Omega} \times \mathbb{R}^q)^{(n)}$. A variational symmetry group of (11) is a local group of transformations G acting on $M \subset \Omega \times \mathbb{R}^q$ with the following property: let Ω_1 be any subdomain with $\bar{\Omega}_1 \subset \Omega$, f a smooth function on Ω_1 whose graph lies in M , and suppose that for $g \in G$, $f' := g \cdot f$ is a well-defined function on some $\Omega'_1 \subset \Omega$, then

$$\int_{\Omega'_1} L(x', \text{pr}^{(n)} f'(x')) dx' = \int_{\Omega_1} L(x, \text{pr}^{(n)} f(x)) dx. \quad (12)$$

Heuristically, therefore, a variational symmetry group is a transformation group that does not change the variational integral. Since in the generalized setting we will exclusively deal with projectable group actions, let us investigate (12) in the case where G is of the form $(x, u) \mapsto (\Xi_\eta(x), \Psi_\eta(x, u))$, Ξ, Ψ smooth, Ξ_η a (w.l.o.g. orientation preserving) diffeomorphism for each η . Then $x' = \Xi_\eta(x)$ is a coordinate transformation and we can rewrite (12) as

$$\int_{\Omega_1} L(\Xi_\eta(x), \text{pr}^{(n)} f'(\Xi_\eta(x))) \det D\Xi_\eta(x) dx = \int_{\Omega_1} L(x, \text{pr}^{(n)} f(x)) dx \quad (13)$$

(with $f' = x' \mapsto \Psi_\eta(\Xi_\eta(x'), f'(\Xi_\eta(x')))$), for each Ω_1 as above and η sufficiently small.

Our strategy basically will be to turn (13) into a definition in the generalized setting. Thus we want to allow for G a generalized projectable group action $\Phi \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^{p+q})^{p+q}$ of the form $(\eta, (x, u)) \mapsto (\Xi_\eta(x), \Psi_\eta(x, u))$. Since contrary to the smooth setting, we cannot argue by continuity in η (in fact, we explicitly want to allow e.g. jumps), in order to make (13) well-defined for general $L \in \mathcal{G}_\tau((\bar{\Omega} \times \mathbb{R}^q)^{(n)})$, we henceforth assume that $\Xi \in \mathcal{G}[(-\eta_0, \eta_0) \times \Omega, \Omega]$ for some $\eta_0 > 0$. This guarantees that for each Ω_1 with $\bar{\Omega}_1 \subset \Omega$ there exists some $\eta_1 \in (0, \eta_0)$ such that $\Xi_\varepsilon(\eta, x)$ remains in a fixed compact subset of Ω for all $\eta < \eta_1$ and ε sufficiently small. Therefore, the following definition makes sense:

Definition 4.1 Let $\Phi = (\eta, (x, u)) \mapsto (\Xi_\eta(x), \Psi_\eta(x, u)) \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^{p+q})^{p+q}$ be a generalized (projectable) group action with $\Xi|_{(-\eta_0, \eta_0) \times \Omega} \in \mathcal{G}[(-\eta_0, \eta_0) \times \Omega, \Omega]$ for some $\eta_0 > 0$. Φ is called a

variational symmetry group of the functional (3) with $\mathcal{L} \in \mathcal{G}_\tau((\bar{\Omega} \times \mathbb{R}^q)^{(n)})$ if for every $\Omega_1 \subseteq \bar{\Omega}_1 \subseteq \Omega$ and every $f \in \mathcal{G}(\Omega_1)^q$ we have (with $f' = x' \mapsto \Psi_\eta(\Xi_{-\eta}(x'), f'(\Xi_{-\eta}(x')))$) :

$$\int_{\Omega_1} L(\Xi_\eta(x), \text{pr}^{(n)} f'(\Xi_\eta(x))) \det D\Xi_\eta(x) dx = \int_{\Omega_1} L(x, \text{pr}^{(n)} f(x)) dx \quad \text{in } \mathcal{G}((-\eta_1, \eta_1)) \quad (14)$$

whenever $\eta_1 > 0$ is such that $\Xi_{\varepsilon\eta}(\bar{\Omega}_1)$ remains in a fixed compact subset of Ω for $\eta < \eta_1$ and ε small.

Remark 4.2 (Inheritance properties of variational symmetries for smooth functionals)

Consider a smooth variational problem ((3) with L smooth) and a smooth projectable variational symmetry Φ . Then since composition of smooth functions and c -bounded generalized functions is well-defined, (14) holds for any $f \in \mathcal{G}[\Omega_1, \mathbb{R}^q]$, since it holds on the level of representatives. The analogous statement is true for $f \in \mathcal{G}(\bar{\Omega}_1)^q$ if we assume that both L and G are slowly increasing, uniformly for x in compact sets (to make the respective compositions well-defined).

Remark 4.3 As was mentioned in Remark 3.2, an alternative admissible set over which the variational problem \mathcal{L} can be considered is the set of all c -bounded generalized functions from $\bar{\Omega}$ to \mathbb{R}^q such that all partial derivatives up to order n are also c -bounded. In this case, in order for the transformed function f' to be well-defined we have to suppose that $\Phi^{(n)} \in \mathcal{G}[\mathbb{R} \times \mathbb{R}^{p+q}, (\mathbb{R}^{p+q})^{(n)}]$ (as can readily be verified by direct calculation using the chain rule).

Our next aim is to derive an infinitesimal criterion for variational symmetries. In its proof we will make use of the following lemma.

Lemma 4.4 Let $\Xi \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^p)^p$ be a \mathcal{G} -complete group of transformations acting on $\mathbb{R} \times \mathbb{R}^p$ with infinitesimal generator ξ . Then

$$\frac{d}{d\eta}(\det J(\eta, x)) = \det J(\eta, x) \cdot \text{Div } \xi(\Xi_\eta(x)),$$

where $J(\eta, x)$ is the Jacobian matrix of Ξ with entries

$$J_{ij}(\eta, x) = \frac{\partial}{\partial x^j}(\Xi_\eta^i(x)) \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^p)^p, \quad 1 \leq i, j \leq p.$$

and $\text{Div } \xi = \sum_{i=1}^p D_i \xi_i$ is the total divergence of ξ .

Proof.

Since the proof of the corresponding result in the smooth case given in [42, Theorem 4.12] can not directly be recovered in the generalized setting we give a direct computational argument. We have

$$\frac{d}{d\eta}(\det J(\eta, x)) = \sum_{k=1}^n \sum_{\sigma \in S_n} \text{sgn}(\sigma) J_{1\sigma(1)}(\eta, x) \cdots \frac{d}{d\eta} J_{k\sigma(k)}(\eta, x) \cdots J_{n\sigma(n)}(\eta, x). \quad (15)$$

Here,

$$\frac{d}{d\eta} J_{k\sigma(k)}(\eta, x) = \frac{\partial}{\partial x^{\sigma(k)}}(\xi^k(\Xi_\eta(x))) = \sum_{l=1}^n \frac{\partial \xi^k}{\partial x^l}(\Xi_\eta(x)) J_{l\sigma(k)}(\eta, x).$$

Inserting into (15) we obtain:

$$\begin{aligned} \frac{d}{d\eta}(\det J(\eta, x)) &= \sum_{k=1}^n \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(J_{1\sigma(1)}(\eta, x) \cdots J_{k\sigma(k)}(\eta, x) \cdots J_{n\sigma(n)}(\eta, x) \frac{\partial \xi^k}{\partial x^k}(\Xi_\eta(x)) \right. \\ &\quad \left. + \sum_{l \neq k} J_{1\sigma(1)}(\eta, x) \cdots J_{l\sigma(k)}(\eta, x) \cdots J_{n\sigma(n)}(\eta, x) \frac{\partial \xi^k}{\partial x^l}(\Xi_\eta(x)) \right) \\ &= \det J(\eta, x) \cdot \text{Div } \xi(\Xi_\eta(x)) + 0. \end{aligned}$$

□

Theorem 4.5 (*Infinitesimal criterion*) Let $\Phi \in \tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^{p+q})^{p+q}$ be a generalized, projectable, strictly \mathcal{G} -complete group action of the form given in Definition 4.1. Then Φ is a variational symmetry group of the functional (3) if and only if

$$\text{pr}^{(n)}\mathbf{v}(L) + L \cdot \text{Div } \xi = 0 \quad \text{in } \mathcal{G}((\Omega \times \mathbb{R}^q)^{(n)}), \quad (16)$$

for the infinitesimal generator

$$\mathbf{v} = \sum_{i=1}^p \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \psi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

of Φ ($\xi^i \in \mathcal{G}_\tau(\mathbb{R}^p), \psi_\alpha \in \mathcal{G}_\tau(\mathbb{R}^{p+q})$).

Proof. Suppose first that Φ is a variational symmetry of (3), and let $f \in \mathcal{G}(\Omega_1)^q$, where $\Omega_1 \subset \Omega$. Then there exists some $\eta_1 > 0$ such that for all $\tilde{\eta} \in (-\eta_1, \eta_1)_c^\sim$ and all $\Omega_2 \subseteq \Omega_1$, by (14) we have

$$\int_{\Omega_2} L(\Xi_{\tilde{\eta}}(x), \text{pr}^{(n)} f'(\Xi_{\tilde{\eta}}(x))) \det D\Xi_{\tilde{\eta}}(x) dx = \int_{\Omega_2} L(x, \text{pr}^{(n)} f(x)) dx \quad \text{in } \tilde{\mathbb{R}}. \quad (17)$$

Since this holds for all open subsets Ω_2 of Ω_1 , we may apply [44, Theorem 1] to conclude that for each $\tilde{\eta}$ as above the integrands in (17) agree in $\mathcal{G}((\Omega_1 \times \mathbb{R}^q)^{(n)})$, i.e.,

$$L(\Xi_{\tilde{\eta}}(x), \text{pr}^{(n)} f'(\Xi_{\tilde{\eta}}(x))) \det(J_{\tilde{\eta}}(x)) = L(x, \text{pr}^{(n)} f(x)) \quad \text{in } \mathcal{G}((\Omega_1 \times \mathbb{R}^q)^{(n)}). \quad (18)$$

Suppose now that $(\tilde{x}, \tilde{u}^{(n)})$ is any compactly supported generalized point in $(\Omega_1 \times \mathbb{R}^q)^{(n)}$. Then we may choose for the components of $f \in \mathcal{G}(\Omega_1)^q$ suitable Taylor polynomials with generalized coefficients such that $(\tilde{x}, \tilde{u}^{(n)}) = \text{pr}^{(n)} f(\tilde{x})$. Therefore by definition of the prolonged group action we have

$$L(\text{pr}^{(n)} \Phi_{\tilde{\eta}}(\tilde{x}, \tilde{u}^{(n)})) \det(J_{\tilde{\eta}}(\tilde{x})) = L(\tilde{x}, \tilde{u}^{(n)}).$$

Since this holds for any given $(\tilde{x}, \tilde{u}^{(n)}) \in ((\Omega_1 \times \mathbb{R}^q)^{(n)})_c^\sim$, by the point value characterization of generalized functions we conclude that

$$L(\text{pr}^{(n)} \Phi_\eta(x, u^{(n)})) \det(J_\eta(x)) = L(x, u^{(n)}) \quad \text{in } \mathcal{G}((- \eta_1, \eta_1) \times (\Omega_1 \times \mathbb{R}^q)^{(n)}) \quad (19)$$

Differentiating this with respect to η yields (by Lemma 4.4):

$$(\text{pr}^{(n)}\mathbf{v}(L) + L \text{Div } \xi) \cdot \det(J_\eta(x)) = 0 \quad \text{in } \mathcal{G}((- \eta_1, \eta_1) \times (\Omega_1 \times \mathbb{R}^q)^{(n)}). \quad (20)$$

If we set $\eta = 0$, we obtain (16) on Ω_1 . Since Ω_1 as above was arbitrary, the claim follows on all of Ω .

Conversely, if (16) holds in $\mathcal{G}((\Omega \times \mathbb{R}^q)^{(n)})$ for the infinitesimal generator ξ of Φ then (20) is automatically satisfied for each pair (Ω_1, η_1) as above. Thus we may integrate (20) from 0 to η to obtain (19). Evaluating (19) at the graph of the generalized function $f \in \mathcal{G}(\Omega_1)^q$ (cf. [16, Definition 4.3.9]), again by the point value characterization in \mathcal{G} we obtain (18), and, therefore, (17), from which we conclude that Φ is a generalized variational symmetry group of \mathcal{L} . \square

As a preparation for Nöther's theorem, we briefly recall the prolongation formula in \mathcal{G} from [16, Theorem 4.3.17] (based on [42, Theorem 2.36]): if $\mathbf{v} = \sum_{i=1}^p \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \psi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$ is a \mathcal{G} - n -complete generalized vector field with corresponding projectable group action Φ on $\mathbb{R}^p \times \mathbb{R}^q$ then

$$\text{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^q \sum_J \psi_\alpha^J(x, u^{(n)}) \partial_{u_J^\alpha}, \quad J = (j_1, \dots, j_k), 1 \leq j_k \leq p, 1 \leq k \leq n,$$

where

$$\psi_\alpha^J(x, u^{(n)}) = D_J(\psi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha \in \mathcal{G}_\tau((\mathbb{R}^p \times \mathbb{R}^q)^{(n)}).$$

As in [42, (2.48)], we may introduce the characteristics of \mathbf{v} as

$$Q_\alpha(x, u^{(1)}) := \psi_\alpha(x, u) - \sum_{i=1}^p \xi^i u_i^\alpha \in \mathcal{G}_\tau((\mathbb{R}^p \times \mathbb{R}^q)^{(1)}) \quad 1 \leq \alpha \leq q,$$

and $Q(x, u^{(1)}) := (Q_1, \dots, Q_q)(x, u^{(1)})$. Then $\psi_\alpha^J = D_J Q_\alpha + \sum_{i=1}^p \xi^i u_{J,i}^\alpha$ and

$$\begin{aligned} \text{pr}^{(n)} \mathbf{v} &= \sum_{\alpha=1}^q \sum_J D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^p \xi^i \left(\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_J u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha} \right) \\ &= \text{pr}^{(n)} \mathbf{v}_Q + \sum_{i=1}^p \xi^i D_i, \end{aligned}$$

with D_i the i -th total derivative, and

$$\mathbf{v}_Q = \sum_{\alpha=1}^q Q_\alpha(x, u^{(1)}) \frac{\partial}{\partial u^\alpha}, \quad \text{pr}^{(n)} \mathbf{v}_Q = \sum_{\alpha=1}^q \sum_J D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}.$$

Nöther's theorem provides a relation between variational symmetry groups and conservation laws for the corresponding system of Euler-Lagrange equations. Recall that a conservation law for a system of differential equations $\Delta(x, u^{(n)}) = 0$ is a divergence expression $\text{Div } P = 0$ which vanishes for all solutions $u = f(x)$ of Δ (P is a p -tuple of generalized functions $P_i(x, u^{(n)}) \in \mathcal{G}((\mathbb{R}^{p+q})^{(n)})$, $1 \leq i \leq p$).

Theorem 4.6 (*Nöther's theorem*) *Let Φ be a projectable, strictly \mathcal{G} -complete variational symmetry group of the variational problem (3), and let*

$$\mathbf{v} = \sum_{i=1}^p \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \psi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

be the infinitesimal generator of Φ with characteristics Q_α as above. Then $Q \cdot E(L)$ is a conservation law for the Euler-Lagrange equations $E(L) = 0$, i.e., there exists a p -tuple $P(x, u^{(n)}) = (P_1, \dots, P_p)(x, u^{(n)})$ such that

$$\text{Div } P = Q \cdot E(L) = \sum_{\nu=1}^q Q_\nu E_\nu(L) \quad \text{in } \mathcal{G}_\tau((\bar{\Omega} \times \mathbb{R}^q)^{(n)}).$$

Proof. The proof of the classical case ([42, Theorem 4.29]) carries over verbatim to our setting. \square

For the practically most relevant case of first order variational problems, the explicit form of P is (cf. [42, Cor. 4.30]):

$$P_i = \sum_{\alpha=1}^q \psi_\alpha \frac{\partial L}{\partial u_i^\alpha} + \xi^i L - \sum_{\alpha=1}^q \sum_{j=1}^p \xi^j u_j^\alpha \frac{\partial L}{\partial u_i^\alpha}.$$

5 Applications

5.1 Geodesics as solutions of a variational problem

Let M be a smooth q -dimensional manifold equipped with a generalized Riemannian metric $g \in \mathcal{G}_2^0(M)$ (cf. [29, 16]). We are looking for curves $c \in \mathcal{G}[[0, T], M]$ which minimize the length functional

$$L(c) := \int_0^T |\dot{c}(t)| dt = \int_0^T \sqrt{g(\dot{c}(t), \dot{c}(t))} dt$$

As in the smooth setting (cf., e.g., [21]), since $L(c) \leq \sqrt{2T}\sqrt{E(c)}$ it suffices to find minimizers of the energy functional

$$E(c) := \frac{1}{2} \int_0^T |\dot{c}(t)|^2 dt = \frac{1}{2} \int_0^T g(\dot{c}(t), \dot{c}(t)) dt. \quad (21)$$

In terms of a local coordinate system (x^1, \dots, x^n) , the Euler-Lagrange equations for E read

$$\frac{d^2 c_\varepsilon^k}{dt^2} + \sum_{i,j} \Gamma_{\varepsilon ij}^k \frac{dc_\varepsilon^i}{dt} \frac{dc_\varepsilon^j}{dt} = 0 \quad k = 1, \dots, q \quad (22)$$

with Γ_{ij}^k the Christoffel symbols of g ,

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right),$$

$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, and $(g^{ij}) = (g_{ij})^{-1}$. These are the usual geodesic equations for the metric g , whose derivation from (21) in the Colombeau setting, based on Lemma 3.3, is completely analogous to that in the smooth case ([21]). (22) is a system of ordinary differential equations for the components of c in $\mathcal{G}([0, T])$.

Analogously, for pseudo-Riemannian generalized metrics, (22) is derived as the Euler-Lagrange equation for E by fixing the causal character (cf. [32]) of c .

As a concrete example let us consider the case of impulsive pp-waves (*plane fronted gravitational waves with parallel rays*) ([2, 45, 27]). Here, the metric is given in the form

$$g = f(x, y) \delta(u) du^2 - dudv + dx^2 + dy^2$$

with f smooth and δ the Dirac measure. Using a generic regularization of δ to embed g into the Colombeau algebra, one can show that the corresponding system of geodesic equations is uniquely solvable in $\mathcal{G}[\mathbb{R}]$. Moreover, this unique solution is associated to the physically expected solution (piecewise straight geodesics which are broken and refracted at the shock hypersurface $u = 0$). We refer to the above papers or also to [16, Sec. 5.3] for a detailed exposition.

5.2 Examples from mechanics

Denote by $x(t)$ the position of a particle of mass 1 in \mathbb{R}^n that moves in a potential $V(x)$. According to Hamilton's least action principle, the particle follows the path that minimizes the action functional

$$\mathcal{L}(x) = \int_a^b L(x(t), \dot{x}(t)) dt \quad (23)$$

among all trajectories $y(t)$ such that $y(a) = x(a)$, $y(b) = x(b)$ for whatever points of time $a < b$. Here the Lagrangian is of the form

$$L(t, x, \dot{x}) = \frac{1}{2} |\dot{x}|^2 - V(x) \quad (24)$$

and given by the difference of the kinetic and the potential energy. It is clear that time translations form a variational symmetry for the functional (23). It follows from Nöther's theorem that the total energy

$$P = \frac{1}{2} |\dot{x}|^2 + V(x)$$

is conserved. Particle systems are of interest here, because our approach offers the possibility to allow singular potentials, for example, delta function potentials. Singular potentials may be modelled by elements $V \in \mathcal{G}_\tau(\mathbb{R}^n)$. If $x \in (\mathcal{G}(\mathbb{R}))^n$ minimizes the action functional (23) on every interval $[a, b]$, then it follows from Theorem 3.4 that x satisfies the Euler-Lagrange-equations

$$\ddot{x}(t) + \nabla V(x) = 0. \quad (25)$$

It has been shown in [16] that (25), completed by initial data $x(t_0) = \tilde{x}_0$, $\dot{x}(t_0) = \tilde{y}_0$ has a solution $x \in (\mathcal{G}(\mathbb{R}))^n$, provided V is either of L^∞ -type or nonnegative. In addition, total energy is conserved along the generalized trajectories.

Uniqueness of the solution to (25) in $(\mathcal{G}(\mathbb{R}))^n$ can be obtained when $\nabla^2 V$ is of L^∞ -log-type. As in the classical case, the second variation of (23) is rarely positive so that uniqueness cannot be inferred using the variational principle alone, in general.

We now turn to a more explicit example – a classical particle in a delta function potential. Formally, it is described by the Lagrangian $L(t, x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 - \delta(x)$. The delta function potential is considered as acting as a barrier so that the particle trajectories exhibit pure reflection at the origin. To make things more precise, let us call a generalized function $D \in \mathcal{G}_\tau(\mathbb{R})$ a *strict delta function* if it has a representative $(\rho_\varepsilon)_\varepsilon \in \mathcal{E}_\tau(\mathbb{R})$ such that

- (i) $\text{supp } \rho_\varepsilon \rightarrow \{0\}$ for $\varepsilon \rightarrow 0$.
- (ii) $\lim_{\varepsilon \rightarrow 0} \int \rho_\varepsilon(x) dx = 1$.
- (iii) $\int |\rho_\varepsilon(x)| dx$ is bounded uniformly in ε .

D is called a *model delta function* if it has a representative $(\varphi_\varepsilon)_\varepsilon$ of the form $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi(\frac{x}{\varepsilon})$, where $\varphi \in \mathcal{D}(\mathbb{R})$, $\int \varphi(x) dx = 1$.

Example 5.1 (Particle in a delta function potential) The Euler-Lagrange equations are

$$\begin{aligned} \ddot{x}(t) + D'(x(t)) &= 0 \\ x(0) &= \tilde{x}_0 \\ \dot{x}(0) &= \tilde{y}_0 \end{aligned} \tag{26}$$

where $D \in \mathcal{G}_\tau(\mathbb{R})$ is a strict or model delta function and $\tilde{x}_0, \tilde{y}_0 \in \tilde{\mathbb{R}}$. We have the following result:

- (i) Problem (26) has a solution $x \in \mathcal{G}(\mathbb{R})$.
- (ii) If D'' is of L^∞ -log-type, the solution is unique in $\mathcal{G}(\mathbb{R})$.
- (iii) If D is a model delta function and $\tilde{x}_0 = x_0, \tilde{y}_0 = y_0$ are elements of \mathbb{R} with $x_0 \neq 0$, then the solution $x \in \mathcal{G}(\mathbb{R})$ of (26) is associated with the function $t \mapsto \text{sign}(x_0)|x_0 + ty_0|$.

The pure reflection picture does not necessarily apply when strict delta functions are employed. In general, it is possible that certain trajectories terminate at the origin, so that the particle may be trapped there after finite time. More details and proofs of these results can be found in [16].

Example 5.2 (Planar motion in a central force field) Introducing polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$ in the plane, the motion of a particle of mass m in the time interval $[a, b]$ is a minimizer of the action functional

$$\mathcal{L}(r, \varphi) = \int_a^b L(r(t), \dot{r}(t), \varphi(t), \dot{\varphi}(t)) dt$$

with

$$L(r, \dot{r}, \varphi, \dot{\varphi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r). \tag{27}$$

Here the potential energy $V(r)$ is assumed to be an element of $\mathcal{G}_\tau(\mathbb{R})$. Such a situation could arise from distributional potentials as in the previous example, or from regularizing classical, non-smooth potentials like the gravitational potential $-1/r$. It is clear that the infinitesimal generator of the rotation

$$\mathbf{v} = \frac{\partial}{\partial \varphi}$$

determines a variational symmetry. According to Nöther's theorem (Theorem 4.6) the corresponding conserved quantity is angular momentum

$$P = \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \dot{\varphi},$$

and this holds for generalized solutions $r, \varphi \in \mathcal{G}[a, b]$ as well.

5.3 Examples from elastostatics

We begin with an example from classical elastostatics. To keep matters simple, we do not treat the full three-dimensional problem but rather focus on a model problem that describes, e. g., vertical displacements of membranes ($n = 2$) or strings ($n = 1$).

Example 5.3 (Standard linear elasticity) Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and connected set with smooth boundary $\partial\Omega$, the reference configuration of an elastic body. We denote by $u : \bar{\Omega} \rightarrow \mathbb{R}$ the displacement. For simplicity, we address the Dirichlet problem $u|_{\partial\Omega} = u_0$ only. The static solution minimizes the functional

$$\mathcal{L}(u) = \frac{1}{2} \int_{\Omega} (\alpha(x)|\nabla u(x)|^2 + \beta(x)u(x)^2) dx - \int_{\Omega} \gamma(x)u(x) dx$$

under all displacements that satisfy the boundary condition. Here $\alpha(x)$ depends on the elastic properties of the material (modulus of elasticity and Poisson's ratio), $\beta(x)$ is a potential that could arise by an elastic constraint, for example, and $\gamma(x)$ is the body force. In our setting, we may take α , β and γ as generalized functions. In this way, singular potentials and degenerate coefficients $\alpha(x)$ can be modelled. A delta potential $\beta(x) = \delta(x - x_0)$ would arise if a spring is attached to a membrane at point $x_0 \in \Omega$. If the coefficient $\alpha(x)$ vanishes in a subregion of Ω , the classical problem loses wellposedness, but the generalized problem might still be wellposed if $\alpha(x)$ is modelled by a generalized function associated with zero, but still invertible.

As in Example 3.8, we make the following assumptions: $\alpha, \beta, \gamma \in \mathcal{G}(\bar{\Omega})$, $\alpha \geq 0$ and invertible on $\bar{\Omega}$, $\beta \geq 0$, $u_0 \in \mathcal{G}(\partial\Omega)$. The admissible set and the admissible variations, respectively, are given by

$$\mathcal{A} = \{u \in \mathcal{G}(\bar{\Omega}) : u|_{\partial\Omega} = u_0\}, \quad \mathcal{A}_0 = \{u \in \mathcal{G}(\bar{\Omega}) : u|_{\partial\Omega} = 0\}.$$

Observe that \mathcal{L} is the quadratic functional

$$\mathcal{L}(u) = \frac{1}{2}a(u, u) - f(u)$$

corresponding to the bilinear form

$$a(u, v) = \int_{\Omega} (\alpha(x)\nabla u(x) \cdot \nabla v(x) + \beta(x)u(x)v(x)) dx$$

and the linear form

$$f(u) = \int_{\Omega} \gamma(x)u(x) dx.$$

The Euler-Lagrange equation is

$$-\operatorname{div}(\alpha(x)\nabla u(x)) + \beta(x)u(x) = \gamma(x), \quad u|_{\partial\Omega} = u_0. \quad (28)$$

The first and second variation was calculated in Example 3.8 and it was shown that $\mathcal{L}(u)$ admits a unique minimizer. It is clear that $u \in \mathcal{A}$ solves the Euler-Lagrange equation (28) in $\mathcal{G}(\Omega)$ if and only if it solves the variational equation (8). In this way, we have proved that solutions to the Euler-Lagrange equation are unique.

In case the coefficients α, β, γ are classical smooth functions, a proof of existence can be obtained as follows: Fixing a representative $u_{0\varepsilon}$ of u_0 , equation (28) has a unique classical smooth solution u_ε , $\varepsilon \in (0, 1]$, cf. [14, Chap. 6]. The closed graph theorem yields continuous dependence on the boundary data in the \mathcal{C}^∞ -seminorms. This shows that $(u_\varepsilon)_\varepsilon$ is moderate and thus defines a solution $u \in \mathcal{G}(\bar{\Omega})$ to (28). An existence result with generalized coefficients $\alpha \in \mathcal{G}(\bar{\Omega})$, $\beta = \gamma = 0$ has been obtained in [43]. Alternatively, the solution can be constructed by direct methods of the calculus of variations, as have been worked out in the Colombeau setting in [11].

Example 5.4 (Euler-Bernoulli beam with discontinuous coefficient) The equilibrium equation for the deflection $w(x)$, $x \in [0, \ell]$ of an Euler-Bernoulli beam under constant axial force β and distributed load $\gamma(x)$ is

$$\frac{d^2}{dx^2} \left(\alpha(x) \frac{d^2 w(x)}{dx^2} \right) + \beta \frac{d^2 w(x)}{dx^2} - \gamma(x) = 0, \quad x \in [0, \ell]. \quad (29)$$

Here ℓ is the length of the beam. The coefficient α is given by the flexural stiffness EI (E the modulus of elasticity, I the area moment of inertia) and may depend on the position x along the beam, in general. Appropriate boundary conditions for a beam freely supported at both ends are

$$w(0) = 0, \quad w(\ell) = 0, \quad \frac{d^2}{dx^2} w(0) = 0, \quad \frac{d^2}{dx^2} w(\ell) = 0. \quad (30)$$

We are particularly interested in the following cases:

- (a) the coefficient α has jump discontinuities (non-smooth case);
- (b) the coefficient α vanishes in certain points (degenerate case);
- (c) the coefficient α is given by a path of positive noise.

Case (a) is a simple model of abrupt change of material properties (see e.g. [20, 47]); case (b) arises for example, if the beam contains a joint in its interior. Classically, these problems are split into adjacent subproblems with transition conditions. A global formulation in the classical setting – particularly in case (b) – is impossible, because the classical ellipticity condition would be violated. Case (c) is of relevance in stochastic structural mechanics. Since α has to be nonnegative, a lognormally distributed noise process would be a suitable model for random disturbances. Such a process does not exist in the classical theory of stochastic processes, but it does exist in the sense of a Wick exponential [18] or as a Colombeau process [41]. In all three cases the Colombeau setting lends itself as a framework that provides a rigorous global solution concept.

We observe that (29) is the Euler-Lagrange equation corresponding to the quadratic functional

$$\mathcal{L}(u) = \frac{1}{2} \int_0^\ell (\alpha(x) |u''(x)|^2 - \beta |u'(x)|^2) dx - \int_0^\ell \gamma(x) u(x) dx.$$

The second variation is given by

$$\delta^2 \mathcal{L}(u, v) = \frac{1}{2} \int_0^\ell (\alpha(x) |v''(x)|^2 - \beta |v'(x)|^2) dx \quad (31)$$

We assume that there is an invertible element $\alpha_0 \geq 0$ in $\tilde{\mathbb{R}}$ such that $\alpha \geq \alpha_0$ in $\mathcal{G}[0, \ell]$. Clearly, the second variation (31) is positive definite if the axial force β is negative. We are going to show that (31) is positive definite even for small, positive axial forces β , below the first eigenvalue of equation (29). To see this, let $w \in \mathcal{C}^\infty[0, \ell]$, $w(0) = w(\ell) = 0$. Then

$$w(x) = -\frac{x}{\ell} \int_0^\ell (\ell - y) w''(y) dy + \int_0^x (x - y) w''(y) dy.$$

From there we derive the inequality

$$\|w\|_{L^2(0, \ell)}^2 \leq C \|w''\|_{L^2(0, \ell)}^2$$

for some constant $C > 0$ and all smooth w as above. Further,

$$\begin{aligned} - \int_0^\ell |w'(x)|^2 dx &= \int_0^\ell w''(x) w(x) dx \\ &\geq -\frac{1}{2} \int_0^\ell |w''(x)|^2 dx - \frac{1}{2} \int_0^\ell |w(x)|^2 dx \\ &\geq -\frac{1}{2} (1 + C) \int_0^\ell |w''(x)|^2 dx. \end{aligned}$$

From this inequality it follows that (31) is positive definite if $\alpha_0 - \frac{1}{2}(1+C)\beta \geq 0$ in $\mathcal{G}[0, \ell]$. If this expression is invertible, then \mathcal{L} admits a unique minimizer (Prop. 3.6), which is obtained as the solution to the Euler-Lagrange equation (29).

As a concrete application, we are going to work out a global formulation of an Euler-Bernoulli beam with a joint, say at half-length $x = \ell/2$. At a joint, the flexural stiffness, i.e. the coefficient α , vanishes. In the Colombeau setting, this can be modelled as follows. Take a smooth unimodal function ψ vanishing for $|x| \geq 1$ and equal to 1 for $|x| \leq 1/2$, $0 \leq \psi(x) \leq 1$ and let $h \approx 0$ define a nonnegative invertible element of $\tilde{\mathbb{R}}$. Put

$$\alpha_\varepsilon(x) = \alpha \left(1 - (1 - h_\varepsilon) \psi \left(\frac{x - \ell/2}{\varepsilon} \right) \right)$$

where $\alpha \in \mathbb{R}$ is some positive constant. Clearly, $(\alpha_\varepsilon)_\varepsilon$ determines a nonnegative invertible element of $\mathcal{G}([0, \ell])$ which is infinitesimally small near $x = \ell/2$ and otherwise equal to α . We consider the Euler-Bernoulli beam without axial force

$$\frac{d^2}{dx^2} \left(\alpha_\varepsilon(x) \frac{d^2 u_\varepsilon}{dx^2}(x) \right) - \gamma(x) = 0, \quad x \in [0, \ell] \quad (32)$$

where $\gamma \in \mathcal{C}^\infty[0, \ell]$ denotes the distributed load. A representative u_ε of the solution in $\mathcal{G}[0, \ell]$ is readily calculated as

$$u_\varepsilon(x) = \int_0^x \int_0^y \frac{M(z)}{\alpha_\varepsilon(z)} dz dy - \frac{x}{\ell} \int_0^\ell \int_0^y \frac{M(z)}{\alpha_\varepsilon(z)} dz dy$$

where $M(x)$, minus the bending moment, is the solution to $M''(x) = \gamma(x)$, $M(0) = M(\ell) = 0$. Now,

$$\int_0^y \frac{M(z)}{\alpha_\varepsilon(z)} dz = \begin{cases} \int_0^y \frac{M(z)}{\alpha} dz, & 0 \leq y \leq \ell/2 - \varepsilon, \\ \int_0^{\ell/2 - \varepsilon} \frac{M(z)}{\alpha} dz + C_\varepsilon + \int_{\ell/2 + \varepsilon}^y \frac{M(z)}{\alpha} dz, & \ell/2 + \varepsilon \leq y \leq \ell \end{cases}$$

with

$$C_\varepsilon = \int_{\ell/2 - \varepsilon}^{\ell/2 + \varepsilon} \frac{M(z)}{\alpha_\varepsilon(z)} dz.$$

If we choose h_ε in such a way that

$$\int_{\ell/2 - \varepsilon}^{\ell/2 + \varepsilon} \frac{dz}{1 - (1 - h_\varepsilon) \psi \left(\frac{z - \ell/2}{\varepsilon} \right)} = \int_{-1}^1 \frac{\varepsilon dy}{1 - (1 - h_\varepsilon) \psi(y)} \rightarrow D \quad (33)$$

in \mathbb{R} as $\varepsilon \rightarrow 0$, we will have that

$$\lim_{\varepsilon \rightarrow 0} \int_0^y \frac{M(z)}{\alpha_\varepsilon(z)} dz = \begin{cases} \int_0^y \frac{M(z)}{\alpha} dz, & 0 \leq y \leq \ell/2, \\ \int_0^{\ell/2} \frac{M(z)}{\alpha} dz + \alpha D M(\frac{\ell}{2}) + \int_{\ell/2}^y \frac{M(z)}{\alpha} dz, & \ell/2 \leq y \leq \ell \end{cases}$$

and so $u_\varepsilon(x)$ will converge to a continuous, piecewise smooth function $u(x)$. This limit – the associated distribution – describes the displacement curve of the beam. Larger values of D correspond to larger loss of stiffness at $x = \ell/2$ and thus to larger displacements of center of the beam. Failure of the beam can be modelled by letting $D = \infty$ in (33).

We remark that the case of a discontinuous flexural stiffness and nonzero axial force has been treated in [20]. Finally, in the stochastic case, positive noise on $[0, \ell]$ can be defined in the Colombeau sense as a lognormally distributed process with mean value 1 and variance given by $\exp(\|\rho_\varepsilon\|_{L^2(\mathbb{R})}^2) - 1$, where ρ_ε is a strict delta function as in Subsection 5.2. The paths of this generalized random process are nonnegative elements of the Colombeau algebra $\mathcal{G}[0, \ell]$ and thus may serve as describing highly random behavior of the flexural stiffness α . We refer to [41] for details on and explanations of the positive noise model.

Example 5.5 (Rods with generalized stress-strain relation; hard rods) Let $u(x)$ be the displacement of a rod of length ℓ of cross-sectional area one, subject to the body force (density) $f(x)$, $0 \leq x \leq \ell$. The balance law is

$$\sigma'(x) + f(x) = 0$$

where σ denotes stress. Letting $\varepsilon = \frac{\partial u}{\partial x}$ the strain, assume a stress-strain relation (constitutive law) of the form $\sigma = g(\varepsilon)$. If the rod is clamped at one end and free at the other, the displacement is the solution to the problem

$$\frac{d}{dx} g\left(\frac{d}{dx} u(x)\right) + f(x) = 0, \quad u(0) = 0, \quad g\left(\frac{d}{dx} u(\ell)\right) = 0. \quad (34)$$

If the constitutive law has a potential, $g(y) = -G'(y)$, equation (34) is the Euler-Lagrange equation of the functional

$$\mathcal{L}(u) = \int_0^\ell \left(G(u'(x)) + f(x)u(x) \right) dx. \quad (35)$$

We allow generalized potentials $G \in \mathcal{G}_\tau[0, \ell]$ and forces $f \in \mathcal{G}[0, \ell]$. The admissible set and admissible variations are

$$\mathcal{A} = \mathcal{A}_0 = \{u \in \mathcal{G}[0, \ell] : u(0) = 0\}.$$

Assume that $u \in \mathcal{A}$ minimizes the functional (35). The first variation is

$$\begin{aligned} \delta \mathcal{L}(u; v) &= \int_0^\ell \left(-g(u'(x))v'(x) + f(x)v(x) \right) dx \\ &= \int_0^\ell \left(g(u'(x))'v(x) + f(x)v(x) \right) dx + g(u'(\ell))v(\ell). \end{aligned}$$

If $\delta \mathcal{L}(u; v) = 0$ for all $v \in \mathcal{A}_0$, we have in particular that

$$g(u'(x))' + f(x) = 0$$

in $\mathcal{G}(0, \ell)$, by using Lemma 3.3. In order to show that $g(u'(\ell)) = 0$ we need an adaptation of the proof of this lemma. Thus assume that $g(u'(\ell)) \neq 0$ in \mathbb{R} . We can find $(r_\varepsilon)_\varepsilon \in \mathbb{R}_M$ and a subsequence $\varepsilon_k \rightarrow 0$ such that $g_{\varepsilon_k}(u'_{\varepsilon_k}(\ell)) r_{\varepsilon_k} = 1$ for all $k \in \mathbb{N}$. On the other hand, there is $N \geq 0$ such that

$$\sup_{0 \leq x \leq \ell} |g_\varepsilon(u'_\varepsilon(x))' + f_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \rightarrow 0.$$

Similar to the argument in Lemma 3.3, we can find an element $v \in \mathcal{G}[0, \ell]$ with support in $(0, \ell]$ such that $v_{\varepsilon_k}(\ell) = r_{\varepsilon_k}$ and

$$\left| \int_0^\ell (g_\varepsilon(u'_\varepsilon(x)) + f_\varepsilon(x))v_\varepsilon(x) dx \right| \leq \frac{1}{2}$$

for all ε , contradicting the hypothesis that $\delta \mathcal{L}(u; v) = 0$ for all $v \in \mathcal{A}_0$. We arrive at the conclusion that a minimizer of (35) in \mathcal{A} is a solution to the Euler-Lagrange equation (34).

In linear elasticity, the stress-strain relation is $\sigma = E\varepsilon$ with the modulus of elasticity E . The larger E , the harder is the rod. In the limiting case $E \rightarrow \infty$, the rod becomes inextensible. We can model such a rod in the Colombeau framework by introducing the potential $G \in \mathcal{G}_\tau(\mathbb{R})$ by $G_\varepsilon(y) = -\frac{1}{\varepsilon}y$. The Euler-Lagrange equation becomes

$$\frac{1}{\varepsilon} u''_\varepsilon(x) + f_\varepsilon(x) = 0, \quad u_\varepsilon(0) = 0, \quad u'_\varepsilon(\ell) = 0$$

and has the solution

$$u_\varepsilon(x) = \varepsilon \int_0^x \int_y f_\varepsilon(z) dz dy.$$

For bounded body force, the solution is associated with zero, thus no extension of the rod takes place, as anticipated. We refer to [33] for the corresponding model in convex analysis.

5.4 Nonlinear wave equations with singular potential

Example 5.6 (Wave equation with delta potential) In this subsection, we discuss the one-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x, t) - \frac{\partial^2}{\partial x^2}u(x, t) + \frac{\partial}{\partial u}V(x, u(x, t)) = 0 \quad (36)$$

with potential $V(x, u)$. The higher dimensional case can be treated similarly. Classical examples of the potential are $V(u) = \frac{m}{2}u^2 + \frac{k}{2}u^4$, $m, k > 0$, leading to the cubic Klein-Gordon equation, or $V(u) = -\cos u$, leading to the Sine-Gordon equation. However, much more singular potentials are in use. For example, the equation

$$\frac{\partial^2}{\partial t^2}u(x, t) - \frac{\partial^2}{\partial x^2}u(x, t) + \delta(x - x_0)F(u(x, t)) = 0 \quad (37)$$

describes the vibrations of a string with a nonlinear spring with restoring force $F(u)$ attached at the point $x = x_0$, see [22]. An even more singular potential,

$$\frac{\partial^2}{\partial t^2}u(x, t) - \frac{\partial^2}{\partial x^2}u(x, t) - \sum_{k=1}^n \delta(u(x, t) - u_k) = 0$$

where $\{u_1, \dots, u_n\}$ is a finite subset of \mathbb{R} , has been studied in [3]. This equation arises from a piecewise constant approximation of a smooth potential – in the quoted reference, the Ginzburg-Landau potential. Note that in this example, the sought after solution has to be inserted in a sum of Dirac measures. All these cases can be subsumed by letting the potential $V(x, u)$ belong to the Colombeau algebra $\tilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R})$. The Lagrangian functional corresponding to (36) is

$$\mathcal{L}(u) = \int_a^b \int_c^d \left(\frac{1}{2} \left| \frac{\partial}{\partial t} u(x, t) \right|^2 - \frac{1}{2} \left| \frac{\partial}{\partial x} u(x, t) \right|^2 - V(x, u(x, t)) \right) dx dt.$$

The element $u \in \mathcal{G}(\mathbb{R}^2)$ that makes $\mathcal{L}(u)$ stationary on every rectangle $[a, b] \times [c, d]$ produces a solution of the Euler-Lagrange equation (36).

Existence and uniqueness of solutions in $\mathcal{G}(\mathbb{R}^2)$ to the Cauchy problem for equation (36) have been demonstrated under various conditions on the potential V , see e.g. [37, 34]. Solutions to equation (37) in the Colombeau algebra $\mathcal{G}(\mathbb{R}^2)$ and their associated distributions have been computed in [38].

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